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Entropy Unilateral Solution for Some Noncoercive Nonlinear Parabolic Problems Via a Sequence of Penalized Equations

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Abstract. We give an existence result of the obstacle parabolic equations

$$\frac{\partial b(x,u)}{\partial t}-div(a(x,t,u,\nabla u))+div(\phi(x,t,u))=f \quad \text{in } Q_T,$$

where $b(x,u)$ is bounded function of u, the term $-\mathrm{div}(a(x,t,u,\nabla u))$ is a Leray-Lions type operator and the function ϕ is a nonlinear lower order and satisfy only the growth condition. The second term f belongs to $L^1(Q_T)$. The proof of an existence solution is based on the penalization methods.

Key Words: Obstacle parabolic problems, entropy solutions, penalization methods.

AMS Subject Classifications: 47A15, 46A32, 47D20

1 Introduction

In this paper, we investigate the problem of existence of solutions of the obstacle problems associated to the following nonlinear parabolic problem:

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t}-div(a(x,t,u,\nabla u))+div(\phi(x,t,u))=f & \text{in } Q_T, \\ u(x,t)=0 & \text{on } \partial\Omega\times(0,T), \\ b(x,u)(t=0)=b(x,u_0(x)) & \text{in } \Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded open set of IR^N $(N\geq 2)$, T is a positive real number, and $Q_T=\Omega\times(0,T)$. Let $b:\Omega\times IR\longrightarrow IR$ is a Carathéodory function such that for every $x\in\Omega$, $b(x,\cdot)$ is a strictly increasing C^1-function, the data f and $b(\cdot,u_0)$ in $L^1(Q_T)$ and $L^1(\Omega)$ respectively. The term $-div(a(x,t,u,\nabla u))$ is a Leray-Lions operator defined on $L^p(0,T;W_0^{1,p}(\Omega))$ (see assumptions (3.3a)-(3.3c)). The function $\phi(x,t,u)$ is a Carathéodory assumed to be continuous on u (see assumptions (3.3d)-(3.3e)). Under these assumptions, the above problem does not admit, in general, a weak solution since the fields $a(x,t,u,\nabla u)$ and $\phi(x,t,u)$ does not belongs in $(L^1_{loc}(Q))^N$ in general.

In the case of equation in the classical Sobolev spaces H. Redwane [5] proved the existence of solution of problem (1.1) where $\phi(x,t,u)=0$, and where $div(\phi(x,t,u))=H(x,t,u,\nabla u)$ and $f\in L^1(Q)$ by Y. Akdim et al. [2] in the degenerated Sobolev spaces without the sign condition and the coercivity condition on the term $H(x,t,u,\nabla u)$.

The existence of a solution is shown in [5, 8] with $b(x,u)=u$, using the framework of renormalized solution, and in [7] for the case $-div(a(x,t,u,\nabla u))=-\Delta u$, using the framework of entropy solution.

It is our purpose, in this paper to generalize the result of [2, 7], and we prove the existence of unilateral entropy solution for the problem (1.1) and without the coercivity condition on ϕ. More precisely, this paper deals with the existence of a solution to the obstacle parabolic problem associated to (1.1) in the sense of unilateral entropy solution (see Theorem 3.1).

The aim of this work is to investigate the relationship between the obstacle problem (1.1) and some penalized sequence of approximate equations (3.9). We study the possibility to find a solution of (1.1) (see Theorem 3.1) as limit of a subsequence u_ϵ of solutions of (3.9). The penalized term $\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^-$ introduced in (3.9) play a crucial role in the proof of our main result, in particular term allows to prove that the solution of (3.9) belongs in the convex set K_ψ.

The plan of the paper is as follows: in Section 2 we give some preliminaries and basic assumptions. In Section 3 we give the definition of entropy solution of (1.1), and we establish (see Theorem 3.1) the existence of such solution.

2 Preliminaries

Let Ω be a bounded open set of IR^N $(N\geq 2)$, T is a positive real number, and $Q_T=\Omega\times(0,T)$. We need the Sobolev embeddings result.

Lemma 2.1 (Gagliardo-Niremberg). *Let $v\in L^q(0,T;L^q(\Omega))\cap L^\infty(0,T;L^\rho(\Omega))$, with $q\geq 1$ and $\rho\geq 1$. Then $v\in L^\sigma(\Omega)$ with $\sigma=q(\frac{N+\rho}{N})$ and*

$$\int_{Q_T}|v|^\sigma dxdt\leq C\|v\|^{\frac{\rho q}{N}}_{L^\infty(0,T;L^\rho(\Omega))}\int_{Q_T}|\nabla v|^q dxdt.$$