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A Characterization of MRA Based Wavelet Frames Generated by the Walsh Polynomials

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Abstract. Extension Principles play a significant role in the construction of MRA based wavelet frames and have attracted much attention for their potential applications in various scientific fields. A novel and simple procedure for the construction of tight wavelet frames generated by the Walsh polynomials using Extension Principles was recently considered by Shah in [Tight wavelet frames generated by the Walsh polynomials, Int. J. Wavelets, Multiresolut. Inf. Process., 11(6) (2013), 1350042]. In this paper, we establish a complete characterization of tight wavelet frames generated by the Walsh polynomials in terms of the polyphase matrices formed by the polyphase components of the Walsh polynomials.

Key Words: Frame, wavelet frame, polyphase matrix, extension principles, Walsh polynomial, Walsh-Fourier transform.

AMS Subject Classifications: 42C15, 42C40, 42A38, 41A17, 22B99

1 Introduction

The most common method to construct tight wavelet frames relies on the so-called Unitary Extension Principles (UEP) introduced by Ron and Shen [11] and were subsequently extended by Daubechies et al. [2] in the form of the Oblique Extension Principle (OEP). They give sufficient conditions for constructing tight and dual wavelet frames for any given refinable function $\phi(x)$, which generates a multiresolution analysis. The resulting wavelet frames are based on multiresolution analysis, and the generators are often called *framelets*. These methods of construction of wavelet frames are generalized from one-dimension to higher-dimension, tight frames to dual frames, from single scaling function to a scaling function vector. Moreover, these principles are important because they can be

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used to construct wavelets from refinable functions which may not be scaling functions with desirable properties such as symmetry and antisymmetry, smoothness or compact support. To mention only a few references on tight wavelet frames, the reader is referred to [4, 6, 9].

The past decade has also witnessed a tremendous interest in the problem of constructing compactly supported orthonormal scaling functions and wavelets with an arbitrary dilation factor $p \geq 2$, $p \in \mathbb{N}$ (see Debnath and Shah [3]). The motivation comes partly from signal processing and numerical applications, where such wavelets are useful in image compression and feature extraction because of their small support and multifractal structure. Lang [10] constructed several examples of compactly supported wavelets for the Cantor dyadic group by following the procedure of Daubechies [1] via scaling filters and these wavelets turn out to be certain lacunary Walsh series on the real line. Kozyrev [8] found a compactly supported p-adic wavelet basis for $L^2(\mathbb{Q}_p)$ which is an analog of the Haar basis. The concept of multiresolution analysis on a positive half-line $\mathbb{R}^+$ was recently introduced by Farkov [5]. He pointed out a method for constructing compactly supported orthogonal p-wavelets related to the Walsh functions, and proved necessary and sufficient conditions for scaling filters with p^n many terms $(p,n \geq 2)$ to generate a p-MRA in $L^2(\mathbb{R}^+)$. Subsequently, dyadic wavelet frames on the positive half-line $\mathbb{R}^+$ were constructed by Shah and Debnath in [17] using the machinery of Walsh-Fourier transforms. They have established a necessary and sufficient conditions for the system $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x \ominus k) : j \in \mathbb{Z},\ k \in \mathbb{Z}^+\}$ to be a frame for $L^2(\mathbb{R}^+)$. Wavelet packets and wavelet frame packets related to the Walsh polynomials were deeply investigated in a series of papers by the author in [13, 14, 18]. Recent results in this direction can also be found in [6, 16] and the references therein.

The second author of this article wrote an article [15] that focuses on the construction of tight wavelet frames generated by the Walsh polynomials on a half-line $\mathbb{R}^+$ based on the ideas from unitary extension principles. More precisely, the author provide a sufficient condition for finite number of functions $\{\psi_1, \psi_2, \cdots, \psi_L\}$ to form a tight wavelet frame for $L^2(\mathbb{R}^+)$ and established a characterization of tight wavelet frames on a positive half-line $\mathbb{R}^+$ by virtue of the modulation matrix $\mathcal{M}(\xi) = \{h_\ell(\xi + k/p)\}_{\ell,k=0}^{L,p-1}$ formed by the Walsh polynomials $h_\ell(\xi)$, $\ell = 0, 1, \cdots, L$ associated with the scaling function $\phi(x)$ and basic wavelets $\psi_\ell(x)$, $\ell = 1, \cdots, L$.

Since the modulation matrix involved in the unitary extension principle has a particular structure and all the information is contained in the first column; the other columns can be derived from the first column by shifting the arguments. On contrary to this, a polyphase matrix is un-structured and this gives an opportunity to create new wavelets from existing ones by multiplying the polyphase matrix by some other appropriate matrix factor. In this paper, we take this opportunity and establish a complete characterization of tight wavelet frames generated by the Walsh polynomials in terms of the polyphase matrix $\Gamma(\xi) = \{\mu_{\ell,r}(\xi)\}_{\ell,r=0}^{p-1}$ formed by the polyphase components $\mu_{\ell,r}(\xi)$, $r = 0, 1, \cdots, p-1$ of the Walsh polynomials $h_\ell(\xi)$.