Analysis in Theory and Applications
Anal. Theory Appl., Vol. **32**, No. 3 (2016), pp. 242-271

DOI: 10.4208/ata.2016.v32.n3.4

Variable Hardy Spaces on the Heisenberg Group

Jingxuan Fang and Jiman Zhao*

School of Mathematical Sciences, Key Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing Normal University, Beijing 100875, China

Received 25 July 2015; Accepted (in revised version) 28 July 2016

Abstract. We consider Hardy spaces with variable exponents defined by grand maximal function on the Heisenberg group. Then we introduce some equivalent characterizations of variable Hardy spaces. By using atomic decomposition and molecular decomposition we get the boundedness of singular integral operators on variable Hardy spaces. We investigate the Littlewood-Paley characterization by virtue of the boundedness of singular integral operators.

Key Words: Hardy spaces, variable exponents, Heisenberg group, atomic decomposition, Littlewood-Paley characterization.

AMS Subject Classifications: 42B20, 42B25, 42B30, 42B35, 43A80

1 Introduction

Hardy spaces have a number of applications in harmonic analysis, as well as in control theory and in scattering theory. The classical Hardy spaces H^p can be characterized by maximal functions, atomic decomposition, and Littlewood-Paley decomposition (see [8, 18, 22, 34, 36, 43], etc.). We can see from [5, 13, 14, 24, 31, 33, 39, 54], etc., that there are many further studies about different kinds of Hardy spaces.

Variable Hardy spaces have been well studied by [11, 37, 41, 48, 49, 52, 55, 56], etc.. We refer to [3, 10, 15–17, 46, 47, 50, 51], etc., for some other kinds of variable function spaces and their applications. We can see that increasing attention has been paid to the study of function spaces with variable exponent in harmonic analysis.

The Heisenberg group, denoted by $\mathbb{H}^n$, plays an important role in several branches of mathematics, such as representation theory, partial differential equations, several complex variables and number theory.

However, as far as we know there is no work investigating variable Hardy spaces on the Heisenberg group. Inspired by the studies of Hardy spaces on some kinds of abstract spaces e.g., [4, 6, 23, 25, 27, 30, 38, 45], we turn to consider characterizing variable Hardy

*Corresponding author. *Email address:* `jzhao@bnu.edu.cn` (J. Zhao)

spaces on the Heisenberg group. This is based on [37] but since the Heisenberg group possesses a special geometry structure, it is a kind of non-Abelian groups and the Fourier transform on it is operator-valued, we need more complicated calculations to extend the classical theories on it. Furthermore, because of the properties of the left invariant vector fields on the Heisenberg group, we have to use sub-Laplace operator $\mathcal{L}$ instead of the classical Laplace operator Δ on $\mathbb{R}^n$ and thus the heat kernel is quite different from the one on the Euclidean spaces.

Firstly, we need to use the structure of dyadic cubes on the doubling metric space given by Tuomas Hytönen and Anna Kairema (see [29]) to generalize some basic theory about variable Lebesgue spaces introduced by David V. Cruz-Uribe and Alberto Fiorenza (see [12]). Then we use a lot of analysis tools on stratified groups introduced by G. B. Folland and E. M. Stein (see [19]) and some basic properties of Fourier transform on the Heisenberg group (see the refrences [44] and [35]) to investigate the variable Hardy spaces on it.

This paper is organized as follows. In Section 2, we recall some basic properties of the Heisenberg group shown in [44] and [19] and then give the definition of variable Hardy spaces and atoms on $\mathbb{H}^n$. In Section 3, we first introduce the log-Hölder continuity and decay condition for the variable exponent $p(\cdot)$. Then under these conditons we prove the equivalence of Hardy norms given by maximal functions, i.e.,

$$\|f\|_{H^{p(\cdot)}_{\mathbb{H}^n}} \sim \|M^*_\varphi f\|_{L^{p(\cdot)}_{\mathbb{H}^n}} \sim \|M_\varphi f\|_{L^{p(\cdot)}_{\mathbb{H}^n}},$$

in Theorem 3.2. Then we characterize variable Hardy spaces by heat kernel in Theorem 3.3. In Section 4, we give the equivalent characterization of variable Hardy spaces by atomic decomposition in Theorem 4.4 and Theorem 4.5 by virtue of the conclusions from Section 3 and then we give the boundedness of singular integral operators in Theorem 4.7 and Theorem 4.9 as an application of atomic decomposition. Finally, in Section 5, by the boundedness of singular integral operators we can give the Littlewood-Paley characterization in Theorem 5.2.

2 Preliminary

In this section we first introduce some basic properties of the Heisenberg group (see [44] and [19]) and then give the definition of variable Hardy spaces on the Heisenberg group.

We denote by $\mathbb{H}^n$ the Heisenberg group, which is a Lie group with the underlying manifold $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}=\mathbb{C}^n\times\mathbb{R}$. The multiplication is given by

$$(x,y,t)(u,v,s)=\Big(x+u,y+v,t+s+\frac{1}{2}(u\cdot y-x\cdot v)\Big),$$

where $u\cdot y=\sum_{j=1}^n u_jy_j$. $\mathbb{H}^n$ is an unimodular group, whose Haar measure coincides with the Lebesgue measure of $\mathbb{R}^{2n+1}$. Different from Euclidean spaces, the dilations on $\mathbb{H}^n$ is