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A PRIORI ERROR ESTIMATES FOR FINITE VOLUME ELEMENT APPROXIMATIONS TO SECOND ORDER LINEAR HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, both semidiscrete and completely discrete finite volume element methods (FVEMs) are analyzed for approximating solutions of a class of linear hyperbolic integro-differential equations in a two-dimensional convex polygonal domain. The effect of numerical quadrature is also examined. In the semidiscrete case, optimal error estimates in $L^\infty(L^2)$ and $L^\infty(H^1)$ norms are shown to hold with minimal regularity assumptions on the initial data, whereas quasi-optimal estimate is derived in $L^\infty(L^\infty)$ norm under higher regularity on the data. Based on a second order explicit method in time, a completely discrete scheme is examined and optimal error estimates are established with a mild condition on the space and time discretizing parameters. Finally, some numerical experiments are conducted which confirm the theoretical order of convergence.

Key words. Finite volume element, hyperbolic integro-differential equation, semidiscrete method, numerical quadrature, Ritz-Volterra projection, completely discrete scheme, optimal error estimates.

1. Introduction

In this paper, we discuss and analyze a finite volume element method for approximating solutions to the following class of second order linear hyperbolic integro-differential equations:

$$
\begin{aligned}
u_{tt} - \nabla\cdot\left(\mathcal{A}(x)\nabla u + \int_0^t \mathcal{B}(x,t,s)\nabla u(s)\,ds\right) &= f(x,t) \quad \text{in } \Omega\times J,\\
u(x,t) &= 0 \quad \text{on } \partial\Omega\times J,\\
u(x,0) &= u_0(x) \quad \text{in } \Omega,\\
u_t(x,0) &= u_1(x) \quad \text{in } \Omega,
\end{aligned} \tag{1}
$$

with given functions u_0 and u_1, where $\Omega\subset\mathbb{R}^2$ is a bounded convex polygonal domain, $J=(0,T]$, $T<\infty$, $u_{tt}=\partial^2u/\partial t^2$ and f is a given function defined on the space-time domain $\Omega\times J$. Here, $\mathcal{A}=[a_{ij}(x)]$ and $\mathcal{B}=[b_{ij}(x,t,s)]$ are 2×2 matrices with smooth coefficients. Further, assume that $\mathcal{A}$ is symmetric and uniformly positive definite in $\bar\Omega$. Problems of this kind arise in linear viscoelastic models, specially in the modelling of viscoelastic materials with memory (cf. Renardy *et al.* [23]).

Earlier, the finite volume difference methods which are based on cell centered grids and approximating the derivatives by difference quotients have been proposed and analyzed, see [15] for a survey. Another approach, which we shall follow in this article was formulated in the framework of Petrov-Galerkin finite element method

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using two different grids to define the trial space and test space. This is popularly known finite volume element methods (FVEMs). Here and also in literatures, the trial space consists of C^0-piecewise linear polynomials on the finite element partition $\mathcal{T}_h$ of $\overline{\Omega}$ and the test space is piecewise constants over the control volume $\mathcal{T}_h^*$ to be defined in Section 2. Earlier, the FVEM has been examined by Bank and Rose [3], Cai [4], Chatzipantelidis [8], Li *et al.* [17], Ewing *et al.* [12], etc. for elliptic problems, for parabolic and parabolic type problems by Chou *et al.* [7], Chatzipantelidis *et al.* [9], Ewing *et al.* [13], Sinha *et al.* [25] and for second order wave equations by Kumar *et al.* [16]. For a recent survey on FVEM, see, a review article by Lin *et al.* [19].

For linear elliptic problems, Li *et al.* [17] have established optimal error estimates in H^1 and L^2 norms. More precisely, for L^2 norm the following estimate is derived:

$$\|u - u_h\|_0 \leq Ch^2 \|u\|_{W^{3,p}(\Omega)}, \ p > 1,$$

where u is the exact solution and u_h is the finite volume element approximation of u. Compared to the error analysis of finite element methods, it is observed that this method is optimal in approximation property, but is not optimal with respect to the regularity of the exact solution as for $O(h^2)$ order convergence, the exact solution $u \in H^3$. For convex polygonal domain Ω, it may be difficult to prove H^3 regularity for the solution u. Therefore, an attempt has been made in [12] to establish optimal L^2 error estimate under the assumption that the exact solution $u \in H^2$ and the source term $f \in H^1$. A counter example has also been provided in [12] to show that if $f \in L^2$, then FVE solution may not have optimal error estimates in L^2 norm. The analysis has been extended to parabolic problems in convex polygonal domain in [9] and optimal error estimates have been derived under some compatibility conditions on the initial data. Further, the effect of quadrature, that is, when the L^2 inner product is replaced by numerical quadrature has been analyzed. Subsequently, Ewing *et al.* [13] have employed FVEM for approximating solutions of parabolic integro-differential equations and derived optimal error estimates under $L^\infty(H^3)$ regularity for the exact solution and $L^2(H^3)$ regularity for its time derivative. Then on convex polygonal domain, Sinha *et al.* [25] have examined semidiscrete FVEM and proved optimal error estimates for smooth and non smooth data. The analysis is further generalized to a second order linear wave equation defined on a convex polygonal domain and *a priori* error estimates have been established only for semidiscrete case, see, Kumar *et al.* [16]. Further, the effect of quadrature and maximum norm estimates are proved under some additional conditions on the initial data and the forcing function. In the present article, an attempt has been made to extend the analysis of FVEM to a class of second order linear hyperbolic integro-differential equations in convex polygonal domains with minimal regularity assumptions on the initial data. Moreover, a completely discrete scheme based on a second order explicit method has been analyzed.

In order to put the present investigation into a proper perspective visa-vis earlier results, we discuss, below, the literature for the second order hyperbolic equations. Li et al. [17] have proved an optimal order of convergence in H^1 norm without quadrature using elliptic projection, but the regularity of the exact solution assumed to be higher than the regularity assumed in our results even when $B = 0$ for the problem (1). On a related finite element analysis for the second order hyperbolic equations without quadrature, we refer to Baker [1] and with quadrature, see, Baker and Dougalis [2] and Dupont [11]. Baker and Dougalis [2] have proved optimal order of convergence in $L^\infty(L^2)$ for the semidiscrete finite element scheme, provided the initial displacement $u_0 \in H^5 \cap H_0^1$ and the initial velocity $u_1 \in H^4 \cap H_0^1$.