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STOCHASTIC GALERKIN METHOD FOR CONSTRAINED OPTIMAL CONTROL PROBLEM GOVERNED BY AN ELLIPTIC INTEGRO-DIFFERENTIAL PDE WITH RANDOM COEFFICIENTS

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Abstract. In this paper, a stochastic finite element approximation scheme is developed for an optimal control problem governed by an elliptic integro-differential equation with random coefficients. Different from the well-studied optimal control problems governed by stochastic PDEs, our control problem has the control constraints of obstacle type, which is mostly seen in real applications. We develop the weak formulation for this control and its stochastic finite element approximation scheme. We then obtain necessary and sufficient optimality conditions for the optimal control and the state, which are the base for deriving a priori error estimates of the approximation in our work. Instead of using the infinite dimensional Lagrange multiplier theory, which is currently used in the literature but often difficult to handle inequality control constraints, we use a direct approach by applying the well-known Lions' Lemma to the reduced optimal problem. This approach is shown to be applicable for a wide range of control constraints. Finally numerical examples are presented to illustrate our theoretical results.

Key words. Priori error estimates, stochastic Galerkin method, optimal control problem, integro-differential equation, constraint of obstacle type.

1. Introduction

Optimal control problems governed by partial different equations have been a major research topic in applied mathematics and control theory. Since the milestone work of J.P Lions [33], a great deal of progress has been made in many aspects such as stability, observability and numerical methods, which are too extensive to be mentioned here even very briefly. Finite element approximation of optimal control problems plays a very important role in numerical methods for these problems, and, the finite element approximation of optimal control problems governed by various partial differential equations, either linear or nonlinear, have been much studied in the literature. For optimal control problems governed by the classic PDEs, the optimality conditions and their finite element approximation and a prior error estimates were established long ago, for example, see the early work in [11]. There have been extensive studies on this aspect for such as elliptic equations, parabolic equations, Stokes equations, and Niavoir-Stokes equations. Some of recent progress in this area has been summarized in [20, 27, 31, 35, 43, 46, 56], and the references cited therein. Systematic introductions of the finite element method for PDEs and optimal control problems can be found in, for example, [43, 46, 56]. There also exists an extensive body of studies adaptive finite element methods for various optimal control problems, which is again too extensive to be mentioned here even very briefly. For a recent summary in computational optimal control, we refer our readers to the recent monograph [36].

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Recently, optimal control problems with more complicated state equations have been considered, particularly those with the integro-differential state equations. Integro-differential equations and their control of this nature appear in applications such as heat conduction in materials with memory, population dynamics, and viscous-elasticity; cf., e.g., Friedman and Shinbrot [12], Heard [21], and Renardy, Hrusa, and Nohel [47]. For equations with nonsmooth kernels, we refer to Grimmer and Pritchard [17], Lunardi and Sinestrari [39], and Lorenzi and Sinestrari [38] and references therein. One very important characteristic of all these models is that they all express a conservation of a certain quantity mass momentum in any moment for any subdomain. This in many applications is the most desirable feature of the approximation method when it comes to numerical solution of the corresponding initial boundary value problem. Furthermore finite element methods for parabolic integro-differential equations problems with a smooth kernel have been discussed in, e.g., Cannon and Lin [6], LeRoux and Thomée [30], Lin, Thomée, and Wahlbin [32], Sloan and Thomée [54], Thomée and Zhang [55], and Yanik and Fairweather [61].

Only very recently the finite element approximation of optimal control with the integro-differential state equations has been systematically studied. For example, the finite element method for the optimal control governed by elliptic integral equations and integro-differential equations has been made in [22], in which the a priori and a posteriori error estimations were obtained. For optimal control problems governed by linear parabolic (and quasi-parabolic) integro-differential equations, a priori error estimates of finite element approximation were studied in [51, 52], hyperbolic integro-differential equations [53]. It is, however, much more difficult to study adaptive finite element methods for control problems governed by linear parabolic integro-differential equations.

Uncertainty, such as uncertain parameters, arises in many complex real-world problems of physical and engineering interests. It is well known that these problems can be described by different kinds of stochastic partial differential equations (SPDEs). In recent years, finite element methods for stochastic elliptic and parabolic PDEs (here we mean the equations with stochastic perturbation in their coefficients.) have been a subject of growing interest in the scientific community (see e.g. [1, 2, 8, 50]), which have been widely used to model fluid flows in porous media in many areas, e.g., transport of pollutants in groundwater and oil recovery processes.

The well known Monte Carlo (MC) method is still the most popular method for simulating stochastic elliptic PDEs and dealing with the statistic characteristics of the solution, although it is a rather computationally expensive method (see e.g. [9, 45]) for higher accuracy. Other alternatives to Monte Carlo method have been employed in the field of stochastic mechanics. A popular technique is the perturbation method, cf. [26]. Given certain smoothness conditions, the random functions and operators involved in the differential equation are expanded in a Taylor series about their respective mean values. Another approach is the Neumann expansion series method, e.g. [1]. In this method the inverse of the boundary value problems stochastic operator is approximated by its Neumann series. Based on a spectral representation of the uncertainty, the spectral stochastic finite element method (SSFEM), e.g. [16] was introduced. This method utilized the Karhunen-Loève expansion of correlated random functions, (cf. [37]), and obtaind the solution by a