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Extremal Functions for Trudinger-Moser Type Inequalities in $\mathbb{R}^N$

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Abstract. Let $N\geq 2$, $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, where ω_{N-1} denotes the area of the unit sphere in $\mathbb{R}^N$. In this note, we prove that for any $0<\alpha<\alpha_N$ and any $\beta>0$, the supremum

$$\sup_{u\in W^{1,N}(\mathbb{R}^N),\|u\|_{W^{1,N}(\mathbb{R}^N)}\leq 1}\int_{\mathbb{R}^N}|u|^\beta\Big(e^{\alpha|u|^{\frac{N}{N-1}}}-\sum_{j=0}^{N-2}\frac{\alpha^j}{j!}|u|^{\frac{Nj}{N-1}}\Big)\mathrm{d}x$$

can be attained by some function $u\in W^{1,N}(\mathbb{R}^N)$ with $\|u\|_{W^{1,N}(\mathbb{R}^N)}=1$. Moreover, when $\alpha\geq\alpha_N$, the above supremum is infinity.

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1 Introduction and main results

Let $N\geq 2$ and $\alpha_N = N\omega_{N-1}^{1/(N-1)}$, where ω_{N-1} is the area of the unit sphere in $\mathbb{R}^N$. For any bounded domain $\Omega\subset\mathbb{R}^N$, we denote $W_0^{1,N}(\Omega)$ the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{W_0^{1,N}(\Omega)} = \Big(\int_\Omega|\nabla u|^N\mathrm{d}x\Big)^{1/N}.$$

The classical Trudinger-Moser inequality [1–5] says

$$\sup_{u\in W_0^{1,N}(\Omega),\|u\|_{W_0^{1,N}(\Omega)}\leq 1}\int_\Omega e^{\alpha_N|u|^{N/(N-1)}}\mathrm{d}x<\infty. \tag{1.1}$$

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Define a function $\zeta:\mathbb{N}\times\mathbb{R}\to\mathbb{R}$ by

$$\zeta(N,t)=e^t-\sum_{j=0}^{N-2}\frac{t^j}{j!}=\sum_{j=N-1}^{\infty}\frac{t^j}{j!}.$$

The inequality (1.1) was extended by Cao [6], Panda [7], do Ó [8], Adachi-Tanaka [9] to the whole $\mathbb{R}^N$, namely

$$\sup_{u\in W^{1,N}(\mathbb{R}^N),\|u\|_{W^{1,N}(\mathbb{R}^N)}\le 1}\int_{\mathbb{R}^N}\zeta(N,\alpha|u|^{\frac{N}{N-1}})\mathrm{d}x<\infty,\qquad \forall 0<\alpha<\alpha_N,\tag{1.2}$$

where

$$\|u\|_{W^{1,N}(\mathbb{R}^N)}=\left(\int_{\mathbb{R}^N}(|\nabla u|^N+|u|^N)\mathrm{d}x\right)^{1/N}.$$

The critical case of (1.2), $\alpha=\alpha_N$, was obtained by Ruf [10] and Li-Ruf [11]. Later, using the Young inequality, Adimurthi-Yang [12] provided a very simple proof of the critical Trudinger-Moser inequality in $\mathbb{R}^N$, as well as the singular Trudinger-Moser inequality. One of conclusions in [12] is that the inequality

$$\int_{\mathbb{R}^N}\frac{\zeta(N,\alpha|u|^{\frac{N}{N-1}})}{|x|^\beta}\mathrm{d}x<\infty,\tag{1.3}$$

holds for any $\alpha>0$, $0\le\beta<N$ and any $u\in W^{1,N}(\mathbb{R}^N)$ $(N\ge 2)$.

It was proved by Ruf [10] and Ishiwata [13] that the supremum

$$\sup_{u\in W^{1,2}(\mathbb{R}^2),\|u\|_{W^{1,2}(\mathbb{R}^2)}\le 1}\int_{\mathbb{R}^2}(e^{\alpha u^2}-1)\mathrm{d}x,$$

can be attained when $\alpha^*\le\alpha<4\pi$ for some constant $\alpha^*>0$, and can not be attained when $0<\alpha\ll 1$. In the case $\alpha=\alpha_N$ and $N\ge 3$, the existence of extremal functions for the supremum in (1.2) was obtained by Li-Ruf [11]; while in the case $0<\alpha<\alpha_N$, the existence result was proved by Ishiwata [13].

From now on, we assume $N\ge 2$. In this note, we first prove a Trudinger-Moser type inequality, namely

Theorem 1.1. (i) *For any $\beta>0$, any $\alpha>0$ and any $u\in W^{1,N}(\mathbb{R}^N)$, there holds*

$$\int_{\mathbb{R}^N}|u|^\beta\zeta(N,\alpha|u|^{\frac{N}{N-1}})\mathrm{d}x<\infty.$$

(ii) *For any $\beta>0$ and any $0<\alpha<\alpha_N$, we have*

$$\sup_{u\in W^{1,N}(\mathbb{R}^N),\|u\|_{W^{1,N}(\mathbb{R}^N)}\le 1}\int_{\mathbb{R}^N}|u|^\beta\zeta(N,\alpha|u|^{\frac{N}{N-1}})\mathrm{d}x<\infty.\tag{1.4}$$

(iii) *For any $\beta>0$ and any $\alpha\ge\alpha_N$, the above supremum is infinity.*