JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **30**, No. 1, pp. 84-94

doi: 10.4208/jpde.v30.n1.7
March 2017

Harnack Inequality and Applications for Stochastic Retarded Differential Equations Driven by Fractional Brownian Motion

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Received 3 November 2016; Accepted 20 December 2016

Abstract. In this paper, by using a semimartingale approximation of a fractional stochastic integration, the global Harnack inequalities for stochastic retarded differential equations driven by fractional Brownian motion with Hurst parameter $0<H<1$ are established. As applications, strong Feller property, log-Harnack inequality and entropy-cost inequality are given.

AMS Subject Classifications: 60G15, 60H05, 60H15

Chinese Library Classifications: O211.63

Key Words: Fractional Brownian motion; Harnack inequality; strong Feller property.

1 Introduction

Under a curvature condition, Wang [1] established the following type dimension-free Harnack inequality for diffusion semigroups on a Riemannian manifold M:

$$(P_t f)^\alpha(y) \le (P_t f^\alpha)(x) e^{c(t)\rho(x,y)^2}, \qquad f\ge 0,\ t>0,\ \alpha>1,\ x,y\in M,$$

where $c(t)>0$ is explicitly determined by α and the curvature lower bound. This type of inequality has been studied extensively, see, for example, Aida and Kawabi [2] and Aida and Zhang [3] for infinite dimensional diffusion processes; Wang [4] for stochastic generalized porous media equations; Röckner and Wang [5] for generalizes Mehler semigroup; Abdelhadi et.al. [6] and Wang and Yuan [7] for stochastic functional equation; Liu [8] for stochastic evolution equations with monotone drifts; Ouyang [9] for Ornstein-Uhnelbeck processes and multivalued stochastic evolution equations etc.

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The Harnack inequality has various applications, see for instance, [5,10–12] for strong Feller property and contractivity properties; [2, 3] for short times behaviors of infinite dimensional diffusions; [13–15] for heat kernel estimates and entropy-cost inequalities. [1, 5, 16] established Harnack inequalities using the method of derivative formula. In order to establish the Harnack inequality for diffusions with curvature unbounded below, the approach of coupling and Girsanov transformations are developed in [17]. This method works also for infinite dimensional SPDE provided the noise is additive and non-degenrate, see e.g. [4, 9, 18, 19] for Harnack inequalities for several different classes of SPDE.

On the other hand, one solution for many SDEs is a semimartingale as well a Markov process. However, many objects in real world are not always such processes since they have long-range aftereffects. Since the work of Mandelbrot and Van Ness [20], there has been an increased interest in stochastic models based on the fractional Brownian motion than Brownian motion. A fractional Brownian motion (fBm) of Hurst parameter $H\in(0,1)$ is a centered Gaussian process $B^H=\{B^H(t),t\geq 0\}$ with the covariance function

$$R_H(t,s)=\mathbb{E}(B_t^H B_s^H)=\frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right).$$

When $H=1/2$ the fBm becomes the standard Brownian motion, and the fBm B^H neither is a semimartingale nor a Markov process if $H\neq 1/2$. However, the fBm B^H, $H>1/2$ is a long-memory process and presents an aggregation behavior. The long-memory property make fBm as a potential candidate to model noise in mathematical finance (see [21]); in biology (see [22]); in communication networks (see, for instance [23]); the analysis of global temperature anomaly [24] and electricity markets [25] etc.

Very recently, using derivative formula, the local Harnack inequalities in the sense that $|x-y|$ is bounded by a constant for the following stochastic differential equations

$$\mathrm{d}X(t)=b(X(t))\mathrm{d}t+\mathrm{d}B^H(t),\quad X(0)=x,$$

driven by fractional Brownian motion with Hurst parameter $1/2<H<1$ were established by Fan in [26]. Subsequently, using the approach of coupling and Girsanov transformations to fractional Brownian motion with Hurst parameter $1/2<H<1$, the global Harnack inequalities for the following stochastic differential equations

$$\mathrm{d}X(t)=b(t,X(t))\mathrm{d}t+\sigma(t)\mathrm{d}B^H(t),\quad X(0)=x,$$

driven by fractional Brownian motion with Hurst parameter $1/2<H<1$ were established by Fan in [27]. Furthermore, using Malliavin calculus, Fan [28] established Bismut derivative formulae for the following stochastic differential equations

$$\mathrm{d}X(t)=b(X(t))\mathrm{d}t+\sigma(t)\mathrm{d}B^H(t),\quad X(0)=x,$$

and functional stochastic differential equations

$$\mathrm{d}X(t)=b(X_t)\mathrm{d}t+\sigma(t)\mathrm{d}B^H(t),\quad X_0=\xi,$$