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Regularity of Viscous Solutions for a Degenerate Non-linear Cauchy Problem

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Abstract. We consider the Cauchy problem for a class of nonlinear degenerate parabolic equation with forcing. By using the vanishing viscosity method it is possible to construct a generalized solution. Moreover, this solution is a Lipschitz function on the spatial variable and Hölder continuous with exponent 1/2 on the temporal variable.

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1 Introduction

In this paper we consider the Cauchy problem for nonlinear degenerate parabolic equation

$$u_t = u\Delta u - \gamma|\nabla u|^2 + f(t,u), \qquad (x,t) \in \mathbb{R}^N \times \mathbb{R}_+, \tag{1.1}$$
$$u(x,0) = u_0(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \tag{1.2}$$

where γ is a nonnegative constant. Eq. (1.1) arises in severals applications of biology and physics, see [1,2]. Eq. (1.1) is of degenerate parabolic type: parabolicity it is loss at

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points where $u=0$, see [1,3] for a most detailed description. In [4] a weak solution for the homogeneous equation (1.1) is constructed by using the vanishing viscosity method [5], the regularity of the weak solutions for the homogeneous Cauchy problem (1.1)-(1.2) was studied by the author in [6] and an extension for the inhomogeneous case is given in [7]. In this paper we extend the above results for the inhomogeneous case, this extension is interesting from physical viewpoint, since the Eq. (1.1) is related with non-equilibrium process in porous media due to external forces. We obtain the following main theorem,

Theorem 1.1. *If* $\gamma \geq \sqrt{2N}-1$, $|\nabla(u_0^{1+\frac{\alpha}{2}})| \leq M$, *where* M *is a positive constant such as*

$$\alpha^2+(\gamma+1)\alpha+\frac{N}{2} \leq 0,$$

then the viscosity solutions of the Cauchy problem (1.1)-(1.2) *satisfies*

$$|\nabla(u^{1+\frac{\alpha}{2}})| \leq M. \tag{1.3}$$

We principally followed the ideas of the authors in [6,7], where the particular reaction therm Ku^m was considered.

2 Preliminaries

We begin this section with the definition of solutions in weak sense.

Definition 2.1. *A function* $u \in L^\infty(\Omega)\bigcap L^2_{loc}([0,+\infty);H^1_{Loc}(\mathbb{R}^N))$, *is called a weak solution of* (1.1)-(1.2) *if it satisfies the following conditions:*

(i) $u(x,t) \geq 0$ *, a.e in* Ω.

(ii) $u(x,t)$ *satisfies the following relation*

$$\int_{\mathbb{R}^N} u_0\psi(x,0)\,\mathrm{d}x+\iint_{\Omega}(u\psi_t-u\nabla u\cdot\nabla\psi-(1+\gamma)|\nabla u|^2\psi-f(t,u)\psi)\,\mathrm{d}x\mathrm{d}t=0, \tag{2.1}$$

for any $\psi \in C^{1,1}(\overline{\Omega})$ *with compact support in* $\overline{\Omega}$.

For the construction of a weak solution to the Cauchy problem (1.1)-(1.2), we use the viscosity method: we add the term $\epsilon\Delta u$ in the Eq. (1.1) and we consider the following Cauchy problem

$$u_t=u\Delta u-\gamma|\nabla u|^2+f(t,u)+\epsilon\Delta u, \qquad u\in\Omega, \tag{2.2}$$

$$u(x,0)=u_0(x), \qquad x\in\mathbb{R}^N, \tag{2.3}$$

where $\gamma \geq 0$. The existence of solutions for (2.2)-(2.3) follows by the Maximum principle and vanishing viscosity method ensures the convergence of the weak solutions when $\epsilon \to 0$ to the Cauchy problem (1.1)-(1.2).

Definition 2.2. *The weak solution for the Cauchy problem* (1.1)-(1.2) *constructed by the vanishing viscosity method is called viscosity solution.*