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A Pohožaev Identity and Critical Exponents of Some Complex Hessian Equations

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Abstract. In this paper, we prove some sharp non-existence results for Dirichlet problems of complex Hessian equations. In particular, we consider a complex Monge-Ampère equation which is a local version of the equation of Kähler-Einstein metric. The non-existence results are proved using the Pohožaev method. We also prove existence results for radially symmetric solutions. The main difference of the complex case with the real case is that we don't know if a priori radially symmetric property holds in the complex case.

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1 Introduction

In [1], Tso considered the following real k-Hessian equation:

$$S_k(u_{\alpha\beta})=(-u)^p \text{ on } \Omega, \quad u=0 \text{ on } \partial\Omega. \tag{1.1}$$

Ω denotes a domain inside $\mathbb{R}^d$. k is an integer satisfying $1\le k\le d$. p is a positive real number. $S_k(u_{\alpha\beta})$ denotes the k-th symmetric polynomial of eigenvalues of the Hessian matrix $(u_{\alpha\beta})=\left(\frac{\partial^2 u}{\partial x^\alpha \partial x^\beta}\right)$. The following formula is well known:

$$S_k(u_{\alpha\beta})=\frac{1}{k!}\sum_{1\le i_1,\dots,j_k\le n}\delta^{i_1\dots i_k}_{j_1\dots j_k}u_{i_1j_1}\dots u_{i_kj_k}.$$

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Here we used the generalized Kronecker symbol $\delta^{i_1\ldots i_k}_{j_1\ldots j_k}$, which is equal to the sign of permutation from $\{i_1\ldots i_k\}$ to $\{j_1\ldots j_k\}$ if the two sets of indices are the same or is equal to 0 otherwise. Tso ([1]) proved following result.

Theorem 1.1. ([1]) *Let Ω be a ball and*

$$\tilde{\gamma}(k,d)=\begin{cases}\frac{(d+2)k}{d-2k} & 1\le k<\frac{d}{2},\\ \infty & \frac{d}{2}\le k<d.\end{cases}$$

Then (i) (1.1) *has no negative solution in $C^1(\bar{\Omega})\cap C^4(\Omega)$ when $p\ge\tilde{\gamma}(k,d)$;* (ii) *It admits a negative solution which is radially symmetric and is in $C^2(\bar{\Omega})$ when $0<p<\tilde{\gamma}(k,d)$, p is not equal to k.*

The non-existence result above was proved by the Pohožaev method. In this article, we first generalize Tso's result to case of complex k-Hessian equation. From now on, let B_R denote the ball of radius R in $\mathbb{C}^n$. We consider the following equation

$$S_k(u_{i\bar{j}})=(-u)^p \text{ on } B_R,\quad u=0 \text{ on } \partial B_R. \tag{1.2}$$

where the complex k-Hessian operator $S_k(u_{i\bar{j}})$ is the k-th symmetric polynomial of eigenvalues of the complex Hessian matrix $(u_{i\bar{j}})=\left(\frac{\partial^2 u}{\partial z^i\partial\bar{z}^j}\right)$, or equivalently we have the following formula:

$$S_k(u_{i\bar{j}})=\frac{1}{k!}\sum_{1\le i_1,\ldots,j_k\le n}\delta^{i_1\ldots i_k}_{j_1\ldots j_k}u_{i_1\bar{j}_1}\ldots u_{i_k\bar{j}_k}.$$

Our first result is

Theorem 1.1. *Define $\gamma(k,n)=\frac{(n+1)k}{n-k}=\tilde{\gamma}(k,2n)$. Then* (i) (1.2) *has no nontrivial nonpositive solution in $C^2(\bar{B}_R)\cap C^4(B_R)$ when $p\ge\gamma(k,n)$;* (ii) *It admits a negative solution which is radially symmetric and is in $C^2(\bar{B}_R)$ when $0<p<\gamma(k,n)$ and p is not equal to k.*

Remark 1.1. Notice that in the above statement the restriction on the solution is only negativity instead of k-plurisubharmonicity. By scaling, we get a solution to $S_k(u_{i\bar{j}})=\lambda(-u)^p$ for any $\lambda>0$ if p satisfies the restrictions. When $p=k$, we are with the eigenvalue problem. As in the real Hessian case ([2]), one should be able to show that there exists a $\lambda_1>0$ such that there is a nontrivial nonpositive solution to the equation: $S_k(u_{i\bar{j}})=\lambda_1(-u)^k$. Moreover, the solution should be unique up to scaling.

Remark 1.2. By the work of [3] and [4], the solution to (1.1) is a priori radially symmetric. However, it's not known if all the solutions to (1.2) are radially symmetric. The classical moving plane method for proving radial symmetry works for many classes of real elliptic equations but doesn't seem to work in the complex case (cf. [4]). For the recent study of complex Hessian equations, see [5–7] and the reference therein.