JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **28**, No. 2, pp. 95-119

doi: 10.4208/jpde.v28.n2.1
June 2015

Existence and Uniqueness of the Global Smooth Solution to the Periodic Boundary Value Problem of Fractional Nonlinear Schrödinger System

WEI Gongming* and DONG Jing

College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China.

Received 4 September 2014; Accepted 24 March 2015

Abstract. In this paper, we study a class of coupled fractional nonlinear Schrödinger system with periodic boundary condition. Using Galerkin method, the existence of global smooth solution is obtained. We also prove the uniqueness of the solution.

AMS Subject Classifications: 35A01, 35Q55

Chinese Library Classifications: O175.29

Key Words: Nonlinear Schrödinger system; global smooth solution; Gagliardo-Nirenberg inequality.

1 Introduction

The Schrödinger equation is the fundamental system of physics for describing nonrelativistic quantum mechanical behavior. It is well known that Feynman and Hibbs [1] used path integrals over Brownian paths to derive the standard (non-fractional) nonlinear Schrödinger equation. Recently, Laskin [2,3] showed that the path integral over the Lévy-like quantum mechanical paths allows to develop the generalization of the quantum mechanics. Namely, if the path integral over Brownian trajectories leads to the well known Schrödinger equation, then the path integral over Lévy trajectories leads to the fractional Schrödinger equation. The fractional Schrödinger equation includes the space derivative of order α instead of the second ($\alpha=1$) order space derivative in the standard Schrödinger equation. Laskin [4] showed the Hermiticity of the fractional Hamilton operator and established the parity conservation law. Guo and Xu [5] studied some physical applications of the fractional Schrödinger equation.

*Corresponding author. *Email addresses:* gmweixy@163.com (G. M. Wei), dongjing1666@126.com (J. Dong)

Nonlinear equations aroused the interest of a large number of mathematicians. In paper [6], Guo and Wang, by using Galerkin method, fixed point principle, and sequential approximation method, proved the existence of periodic solutions for the nonlinear systems of Schrödinger equations with Dirichlet boundary conditions and periodic condition in time. Guo [7,8] showed the existence and nonexistence of a global solution for one class of the strong nonlinear Schrödinger equations of fourth order, and studied the initial value problem of integral nonlinear Schrödinger system. In [9] Guo considered a class of high order multidimensional nonlinear systems of Schrödinger equations for the initial boundary value and the periodic initial value problem.

In this paper, we consider the following fractional nonlinear Schrödinger system

$$\begin{cases} iu_t+(-\triangle)^{\alpha}u+\beta|u|^{\rho}u+\zeta v=0, \\ iv_t+(-\triangle)^{\alpha}v+\beta|v|^{\rho}v+\gamma u=0, \end{cases} \tag{1.1}$$

with the following initial condition and periodic boundary condition

$$\begin{cases} u(x,0)=u_0(x), \quad x\in R^n, \\ v(x,0)=v_0(x), \quad x\in R^n, \end{cases} \tag{1.2}$$

and

$$\begin{cases} u(x+2\pi e_i,t)=u(x,t), \quad x\in R^n,t>0, \\ v(x+2\pi e_i,t)=v(x,t), \quad x\in R^n,t>0, \end{cases} \tag{1.3}$$

where $e_i=(0,\dots,0,1,0,\dots,0)$ $(i=1,\dots,n)$ is an orthonormal basis of R^n. In (1.1), i is the imaginary unit, α is a positive fraction, $\gamma,\zeta,\beta\in R$, $\beta\notin 0$, $\gamma,\zeta,\rho>0$.

When $\alpha=1$, (1.1) becomes the standard nonlinear Schrödinger system which has been studied by many works, e.g. [6]. The existence and uniqueness of the weak solution to the initial-boundary value problem can be found in [10]. The existence of the global smooth solution to the initial-boundary value problem was studied in [11,12]. However, there is few mathematical analysis for the fractional nonlinear Schrödinger system (1.1).

In this paper, we prove the existence of the global smooth solution (as smooth in the H^{α}) to the period boundary value problem of fractional nonlinear Schrödinger system (1.1) by using the energy method. Theorems 2.1 and 4.1 are the new fundamental results of the fractional nonlinear Schrödinger system. We obtain respectively the global existence of smooth solution to the period boundary value problem of the fractional nonlinear Schrödinger system in Theorems 2.1 and 4.1.

2 Preliminary and main results

Let u be a periodic function, we can express u by a Fourier series:

$$u=\sum_{k\in Z^n} a_k e^{i\langle k,x\rangle}.$$