JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **26**, No. 3, pp. 226-250

doi: 10.4208/jpde.v26.n3.3
September 2013

Existence, Uniqueness and Blow-Up Rate of Large Solutions of Quasi-Linear Elliptic Equations with Higher Order and Large Perturbation

ZHANG Qihu[1,*] and ZHAO Chunshan[2]

[1] *Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou 450002, China.*
[2] *Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460, USA.*

Received 10 January 2013; Accepted 8 June 2013

Abstract. We establish the existence, uniqueness and the blow-up rate of the large positive solution of the quasi-linear elliptic problem

$$-\triangle_p u = \lambda(x) u^{\theta-1} - b(x) h(u), \quad \text{in } \Omega,$$

with boundary condition $u=+\infty$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a smooth bounded domain, $1<p<\infty$, $\lambda(\cdot)$ and $b(\cdot)$ are positive weight functions and $h(u) \sim u^{q-1}$ as $u \to \infty$. Our results extend the previous work [Z. Xie, *J. Diff. Equ.*, 247 (2009), 344-363] from case $p=2$, λ is a constant and $\theta=2$ to case $1<p<\infty$, λ is a function and $1<\theta<q$ $(q>p)$; and also extends the previous work [Z. Xie, C. Zhao, *J. Diff. Equ.*, 252 (2012), 1776-1788], from case λ is a constant and $\theta=p$ to case λ is a function and $1<\theta<q$ $(>p)$. Moreover, we remove the assumption of radial symmetry of the problem and we do not require $h(\cdot)$ is increasing.

AMS Subject Classifications: 35J25, 35J60

Chinese Library Classifications: O175.25

Key Words: Blow up rate; large positive solution; quasi-linear elliptic problem; uniqueness.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with C^2 smooth boundary $\partial\Omega$. We consider the existence, uniqueness and the blow-up rate of the large solutions of the quasi-linear

*Corresponding author. *Email addresses:* `zhangqh1999@yahoo.com.cn, zhangqihu@yahoo.com` (Q. Zhang), `czhao@GeorgiaSouthern.edu` (C. Zhao)

elliptic problem with singular boundary value condition as follows:

$$-\Delta_p u=\lambda(x)u^{\theta-1}-b(x)h(u), \qquad \text{in } \Omega, \tag{1.1a}$$
$$u\geq 0, \qquad \text{in } \Omega, \tag{1.1b}$$
$$u=+\infty, \qquad \text{on } \partial\Omega, \tag{1.1c}$$

where $\Delta_p u=\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ with $\nabla u(x)=(\partial_{x_1}u,\partial_{x_2}u,\cdots,\partial_{x_N}u)$, $\theta,p\in(1,\infty)$ and $\lambda(\cdot)>0$, $b(\cdot)>0$ are weighted functions which we will explain later, and the boundary condition (1.1c) is understood as $u(x)\to+\infty$ when $d(x):=\operatorname{dist}(x,\partial\Omega)\to 0^+$. The solutions of (1.1a)-(1.1c) are called *large* (or *blow-up*) *solutions*.

We say that $\lambda(x)u^{\theta-1}$ is a perturbation term. If $\theta>p$, we say that $\lambda(x)u^{\theta-1}$ is a higher order perturbation, otherwise, it is called a lower order perturbation. For any solution $u(x)$ of (1.1a)-(1.1c), if

$$\frac{\lambda(x)u^{\theta-1}}{b(x)h(u)}\to 0, \quad \text{as } d(x)\to 0,$$

we say that $\lambda(x)u^{\theta-1}$ is a small perturbation, otherwise it is called a large perturbation.

The p-Laplacian operator

$$\Delta_p u=\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

appears in the study of non-Newtonian flows, chemotaxis, and biological pattern formation etc. When $p=\theta=2$, the problem (1.1a)-(1.1c) becomes as follows:

$$-\Delta u=\lambda(x)u-b(x)h(u), \qquad \text{in } \Omega, \tag{1.2a}$$
$$u\geq 0, \qquad \text{in } \Omega, \tag{1.2b}$$
$$u=\infty, \qquad \text{on } \partial\Omega, \tag{1.2c}$$

and it has been studied extensively. Bieberbach [1] studied the large solutions for the particular case $-\triangle u=-\exp(u)$ with conditions (1.2c) in smooth bounded two-dimensional domains. Later on, Rademacher [2] continued the study of the large solutions for the particular case $-\triangle u=-\exp(u)$ in smooth bounded domains in $\mathbb{R}^3$. Bandle-Essen [3] and Lazer-Mckenna [4] extended Bieberbach's and Rademacher's results to general case $-\triangle u=-b(x)\exp(u)$ in smooth bounded domains of $\mathbb{R}^N$, where the function $b(x)$ is continuous and strictly positive on $\overline{\Omega}$. They showed that the problem has a unique solution together with an estimate of the form $u=\log d^{-2}+o(d)$, see [3] for case $b\equiv 1$ and [4] for case $b(x)\geq b_0>0$ as $d\to 0$.

Recently, the uniqueness of solutions for (1.2a)-(1.2c) with $h(u)=u^q$ $(q>1)$ on bounded domains or the whole space $\mathbb{R}^N$ was discussed in many papers (see, e.g., [2–23]). By using the localization method of [14], it was shown in [14, 19] that (1.2a)-(1.2c) with $h(u)=u^q$, $q>1$ has at most one blow-up solution under some conditions. Further improvements of these results can be found in [5, 6, 15, 18, 20, 22, 23] and the references therein.

The radial case of the problem (1.2a)-(1.2c) on a ball domain $B_R(x_0)$ with $h(u)=u^q$ was firstly studied by López-Gómez [15]. The author also extended the results to a general