JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **25**, No. 1, pp. 1-20

doi: 10.4208/jpde.v25.n1.1
February 2012

Non-Existence of Global Solutions for a Fractional Wave-Diffusion Equation

BERBICHE Mohamed[1] and HAKEM Ali[2,*]

[1] *Département de Mathématiques, Université de Khenchela, Route de Constantine Bp, 1252, El-Houria 40004 Khenchela Algérie.*
[2] *Laboratoire d'Informatique EEDIS, Université de Sidi Bel Abbes, Sidi-Bel-Abbes, Algérie.*

Received 24 September 2009; Accepted 24 November 2011

Abstract. We considered the Cauchy problem for the fractional wave-diffusion equation

$$D^{\alpha}u-\Delta|u|^{m-1}u+(-\Delta)^{\beta/2}D^{\gamma}|u|^{l-1}u=h(x,t)|u|^{p}+f(x,t)$$

with given initial data and where $p>1$, $1<\alpha<2$, $0<\beta<2$, $0<\gamma<1$. Nonexistence results and necessary conditions for global existence are established by means of the test function method. This results extend previous works.

AMS Subject Classifications: 35L20, 35L70, 34A12

Chinese Library Classifications: O175.27

Key Words: Nonlinear wave-diffusion equation; fractional power derivative; critical exponent.

1 Introduction

In [1], Kirane and Tatar consider the Cauchy problem of the hyperbolic fractional equation

$$u_{tt}-\Delta u+D^{\beta}u=h(x,t)|u|^{p}, \tag{1.1}$$

where $p>1$ and $0<\beta<1$, this equation arises in the modeling of fast wave propagation in micro-inhomogeneous media see (see [2]). In [1], the authors established conditions on the initial data and the function $h(x,t)$ that are necessary for local and global existence. It is shown that if

$$1<p\leq 1+\frac{2\beta+\rho}{2+N-2\beta},$$

*Corresponding author. *Email addresses:* `berbichemed@yahoo.fr` (M. Berbiche), `hakemali@yahoo.com` (A. Hakem)

(where ρ comes from the function h) then we have non-existence of global solutions.

When $m=l=1$, $\alpha=2$, $\beta=0$, $h=1$ and $\gamma=1$, this problem has been treated by a large number of researchers. Then we obtain the wave equation with the linear damping u_t. In this case Todorova and Yordanov [3], Mitidieri and Pohozahev [4] and Zhang [5] showed that the Fujita exponent is $p_c=1+2/N$. This result has been extended to solutions of the telegraph equation

$$D^{2\beta}u-\Delta u+D^{\beta}u=0,$$

by Cascaval et al. [6] this problem arises while studying some iterated Brownian motions (see [7]). We point out here that fractional derivatives serve, among other things, to model various anomalous damping such as noise attenuation and viscoelastic dissipations (see [8–12]). Indeed it has been shown by experiments (see [13]) that experiment data fit very well in the models involving fractional derivatives within a broad frequency range for several materials. This materials include synthetic polymers, electrochemistry, glassy materials and many other viscoelastic and hereditary mechanics.

In this paper, we consider the problem

$$\begin{cases} D^{\alpha}u-\Delta|u|^{m-1}u+(-\Delta)^{\beta/2}D^{\gamma}|u|^{l-1}u=h(x,t)|u|^p+f(x,t), \\ u(x,0)=u_0(x)\geq 0,\ u_t(x,0)=u_1(x)\geq 0, \qquad x\in \mathbb{R}^N. \end{cases} \tag{1.2}$$

We will generalize the results in [1] to problem (1.2) where $1<\alpha\leq 2$, $0<\beta<2$ and $0<\gamma<1$. Nonexistence results as well as necessary conditions for local and global existence will be established. In addition to this we can look at the equation in problem (1.2) as a generalization of the fractional diffusion-wave equation

$$D^{\alpha}u=\Delta u+h(x,t)|u|^p, \qquad 1<\alpha\leq 2. \tag{1.3}$$

This eq. (1.3) is now a special case of the eq. (1.2), we can consider the eq. (1.2) as the fractionally damped equation of (1.3). Eq. (1.3) serves as a model in the study of the thermal diffusion in fractal media. See Saichev and Zaslavsky [14], Mainardi [10, 11], Fujita [15] and references therein. Molz et al. in [16] discuss a physical interpretation of the fractional derivative in a Levy diffusion process. Our argument is based on the test-function method developed by Mitidieri and Pohozaev [4], Zhang [5] Kirane and Tatar [1] and others. The necessary conditions results are inspired by some arguments due to Baras and Kersner [17].

Now, we present two different definitions of fractional derivatives (see [13, 18]).

We define the fractional derivative in the Caputo sense of power μ by

$${}^{C}D^{\mu}_{+}u(t):=\frac{1}{\Gamma(n-\mu)}\int_0^t(t-\tau)^{n-\mu-1}u^{(n)}(\tau)\mathrm{d}\tau, \qquad n-1<\mu<n.$$

The fractional derivative in the Riemann-Liouville sense is given by

$${}^{RL}D^{\mu}_{+}u(t):=\frac{1}{\Gamma(n-\mu)}(\frac{\mathrm{d}}{\mathrm{d}t})^n\int_0^t(t-\tau)^{n-\mu-1}u(\tau)\mathrm{d}\tau, \qquad t>0.$$