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Standing Wave Solutions in Nonhomogeneous Delayed Synaptically Coupled Neuronal Networks

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Abstract. The authors establish the existence and stability of standing wave solutions of a nonlinear singularly perturbed system of integral differential equations and a nonlinear scalar integral differential equation. It will be shown that there exist six standing wave solutions $\big((u(x,t),w(x,t)\big)=\big(U(x),W(x)\big)$ to the nonlinear singularly perturbed system of integral differential equations. Similarly, there exist six standing wave solutions $u(x,t)=U(x)$ to the nonlinear scalar integral differential equation. The main idea to establish the stability is to construct Evans functions corresponding to several associated eigenvalue problems.

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1 Introduction

1.1 The mathematical model equations

The authors consider the following nonlinear singularly perturbed system of integral differential equations arising from nonhomogeneous synaptically coupled neuronal networks

$$\frac{\partial u}{\partial t}+f(u)+w=(\alpha-au)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}}K(x-y)H\left(u\left(y,t-\frac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c$$
$$+(\beta-bu)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}}W(x-y)H(u(y,t-\tau)-\Theta)\mathrm{d}y\right]\mathrm{d}\tau+I(x,t), \quad (1.1)$$

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$$\frac{\partial w}{\partial t} = \varepsilon\big[g(u) - w\big], \tag{1.2}$$

where $u = u(x,t)$ represents the membrane potential of a neuron at position x and time t in a nonhomogeneous synaptically coupled neuronal network, $w = w(x,t)$ represents the leaking current. The probability density functions ξ and η are defined on $(0,\infty)$. The kernel functions K and W are defined on $\mathbb{R}$. They represent synaptic couplings between neurons. The function ξ represents a statistical distribution of action potential speeds. Additionally, ξ may have a compact support $[c_1,c_2]$, where c_1 and c_2 are positive constants, denoting the lower bound and upper bound of biologically possible speeds, respectively. Moreover, a, b, α, β, ε, θ and Θ are nonnegative or positive constants, representing various biological mechanisms. The functions $f = f(u)$ and $g = g(u)$ are smooth functions. In addition, either $f(u)+g(u) = m(u-n)+k(u-l)$ is a linear function, where $k>0$ and $m>0$ are positive constants, l and n are real constants; or $f(u)+g(u) = u(u-1)(Du-1)$ is a cubic polynomial function, where $D>0$ is a positive constant. In this model system, we choose the gain function to be the Heaviside step function: $H(u-\theta)=0$ for all $u<\theta$, $H(0)=1/2$, and $H(u-\theta)=1$ for all $u>\theta$. The nonhomogeneous function $I=I(x,t)$ is an externally applied current. See [1–20] for the same or very similar model equations.

If $\varepsilon=0$ and $w=0$ in the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2), then we have a nonlinear scalar integral differential equation

$$\begin{aligned}\frac{\partial u}{\partial t} + f(u) &= (\alpha - au)\int_0^\infty \xi(c)\left[\int_{\mathbb{R}} K(x-y) H\left(u\left(y,t-\frac{1}{c}|x-y|\right)-\theta\right)\mathrm{d}y\right]\mathrm{d}c \\ &\quad + (\beta - bu)\int_0^\infty \eta(\tau)\left[\int_{\mathbb{R}} W(x-y)H(u(y,t-\tau)-\Theta)\mathrm{d}y\right]\mathrm{d}\tau + I(x,t).\end{aligned} \tag{1.3}$$

The authors will study the existence and stability of standing wave solutions of the form $\big(u(x,t),w(x,t)\big) = \big(U(x),W(x)\big)$ to the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2) and study the standing wave solutions of the form $u(x,t)=U(x)$ to the nonlinear scalar integral differential equation (1.3).

The nonlinear system and the scalar equation generalize many important integral differential equations arising from nonhomogeneous synaptically coupled neuronal networks. Previously, Amari [22], Guo and Chow [23], Pinto and Ermentrout [7] and Zhang [14, 21] have studied the existence and stability of standing wave solutions of some nonlinear integral differential equations arising from synaptically coupled neuronal networks. However, the existence and stability of standing wave solutions of the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2) have been open for a long time. An interesting feature on the stability analysis is that the eigenvalue problems derived from linearization of the nonlinear singularly perturbed system of integral differential equations (1.1)-(1.2) are nonlinear in λ (this is the eigenvalue parameter). This difficulty arises because the system involves two kinds of delays and any of the two delays may cause such difficulty. The authors are able to overcome the difficulty to find the eigenvalues of the eigenvalue problems by the construction of Evans functions and by studying their properties.