JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **24**, No. 4, pp. 361-384

doi: 10.4208/jpde.v24.n4.7
November 2011

Well- and Ill-Posedness Issues for a Class of 2D Wave Equation with Exponential Growth

MAHOUACHI Olfa* and SAANOUNI Tarek

Laboratoire d'équations aux dérivées partielles et applications Faculté des Sciences de Tunis, Département de Mathématiques Campus universitaire 1060, Tunis, Tunisie.

Received 18 October 2010; Accepted 13 August 2011

Abstract. Extending the previous work [1], we establish well-posedness results for a more general class of semilinear wave equations with exponential growth. First, we investigate the well-posedness in the energy space. Then, we prove the propagation of the regularity in the Sobolev spaces $H^s(\mathbb{R}^2)$ with $s \geq 1$. Finally, an ill-posedness result is obtained in $H^s(\mathbb{R}^2)$ for $s<1$.

AMS Subject Classifications: 35L70, 35B40, 35B44, 47J30

Chinese Library Classifications: O175.23, O175.27, O176

Key Words: Nonlinear wave equation; well-posedness; ill-posedness; Moser-Trudinger inequality; finite speed of propagation.

1 Introduction

We consider the initial value problem for a semi-linear Klein-Gordon equation

$$\Box u+u+f(u):=\partial_t^2 u-\triangle_x u+u+f(u)=0 \qquad \text{in } \mathbb{R}_t\times\mathbb{R}_x^2, \tag{1.1}$$

with data

$$(u(0,.),\partial_t u(0,.)):=(u_0,u_1)\in H^s(\mathbb{R}^2)\times H^{s-1}(\mathbb{R}^2). \tag{1.2}$$

$H^s:=(1-\triangle)^{-s/2}L^2$ is the inhomogeneous Sobolev space, $u(t,x):\mathbb{R}^+\times\mathbb{R}^2\to\mathbb{R}$ and $f\in C^1(\mathbb{R})$ satisfies the following exponential growth condition:

$$\begin{cases} f(0)=f'(0)=0, \\ \forall\alpha>0,\exists\Re_\alpha>0 \quad \text{s.t.} \quad |f(u)-f(v)|^2\leq\Re_\alpha|u-v|^2\left(e^{\alpha u^2}-1+e^{\alpha v^2}-1\right). \end{cases} \tag{$\mathcal{H}$}$$

*Corresponding author. *Email addresses:* olfa.mahouachi@fst.rnu.tn (O. Mahouachi), Tarek.saanouni@ipeiem.rnu.tn (T. Saanouni)

Solutions to (1.1)-(1.2) formally satisfy the conservation of the energy

$$E(u,t):=\frac{1}{2}\Big(\|u(t)\|^2_{L^2(\mathbb{R}^2)}+\|\partial_t u(t)\|^2_{L^2(\mathbb{R}^2)}+\|\nabla u(t)\|^2_{L^2(\mathbb{R}^2)}\Big)+\int_{\mathbb{R}^2}F(u(t,x))\mathrm{d}x, \tag{1.3}$$

where F is the primitive of f which vanishes on zero.

This paper is concerned with the well and ill-posedness issues of the Cauchy problem (1.1)-(1.2) in the Sobolev space $H^s(\mathbb{R}^2)$. We show local (respectively global) well-posedness of the Cauchy problem (1.1)-(1.2) whenever $s\geq 1$ (respectively in the defocusing case $uf(u)\geq 0$). An instability result holds when $s<1$. Before going further, let recall a few historic facts about this problem. We begin with the monomial defocusing semi-linear wave equation in space dimensions $d\geq 3$,

$$\Box u+|u|^{p-1}u=0, \quad p>1. \tag{1.4}$$

The well-posedness of (1.4) in the scale of the Sobolev spaces H^s has been widely investigated (see for instance [2–7]). It is well-known that the Cauchy problem associated to (1.4) is locally well-posed in the usual Sobolev space $H^s(\mathbb{R}^d)$ if $s>d/2$, or when $1/2\leq s<d/2$ and $p\leq 1+4/(d-2s)$ [8–10]. Moreover if $p=1+4/(d-2s)$ and $1/2\leq s<d/2$, then we have global H^s-solutions for small Cauchy data [9, 11].

The global solvability in the energy space $H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ has attracted a great deal of works. A critical value of the power p appears, namely $p_c:=(d+2)/(d-2)$ and there are mainly three cases. In the subcritical case $(p<p_c)$, Ginibre and Velo proved in [3] the global existence and uniqueness in the energy space.

In the critical case $(p=p_c)$, the global existence was first proved by Struwe in the radially symmetric case [12], then by Grillakis [13] in the general case and later on by Shatah-Struwe [7] in other dimensions.

In the supercritical case $(p>p_c)$, the question remains open except for some partial results (see [14, 15]).

In two space dimensions any polynomial nonlinearity is subcritical with respect to the H^1-norm. Hence, it is legitimate to consider an exponential nonlinearity in two space dimension. Moreover, the choice of an exponential nonlinearity emerges from a possible control of solutions via a Moser-Trudinger type inequality (see Proposition 1.2 and Remark 1.3 below). In fact, Nakamura and Ozawa [11] proved global well-posedness and scattering for small Cauchy data in any space dimension $N\geq 2$. Later on, A. Atallah [16] showed a local existence result to the 2D equation

$$\partial_t^2 u-\Delta_x u+u\mathrm{e}^{\alpha u^2}=0 \qquad (E_\alpha) \tag{1.5}$$

for $0<\alpha<4\pi$ and with radially symmetric initial data $(0,u_1)$ having compact support. Recently, Ibrahim-Majdoub-Masmoudi [17] obtained the global well-posedness in the energy space for both subcritical and critical cases. We infer that the Cauchy problem associated to $(E_{4\pi})$ is said to be subcritical if

$$E^0:=\|u_1\|^2_{L^2(\mathbb{R}^2)}+\|\nabla u_0\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{4\pi}\int_{\mathbb{R}^2}\Big(\mathrm{e}^{4\pi u_0^2}-1\Big)\mathrm{d}x<1.$$