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Regularity of Radial Solutions to the Complex Hessian Equations

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Abstract. In this paper we consider regularities of radial solutions to the degenerate complex Hessian equations. Our results generalize some results for Monge-Ampere equation in [Monn, Math. Ana. 275 (1986), pp. 501-511] and [Delanoe, J. Diff. Eqn. 58 (1985), pp. 318-344].

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1 Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$, and let $u\in C^2(\Omega)$ be a real valued-function. Then the complex Hessian of u defined by

$$[u_{i\bar{j}}]=\left[\frac{\partial^2 u(z)}{\partial z_i\partial \bar{z}_j}\right]$$

is an $n\times n$ Hermitian matrix at each point $z\in\Omega$. Let H_k denote the complex Hessian operator in $\mathbb{C}^n$, which is defined for C^2 functions u as follows:

$$H_k[u]=\sigma_k(u_{i\bar{j}}),\quad 1\leq k\leq n,$$

where σ_k is the k-th elementary symmetric function for the eigenvalues of Hessian matrix $[u_{i\bar{j}}]$. That is, for $1\leq k\leq n$ and $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{R}^n$,

$$\sigma_k(\lambda)=\sum_{i_1<\cdots<i_k}\lambda_{i_1}\cdots\lambda_{i_k},$$

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which coincides with the Laplace $H_1[u]=\Delta u$ if $k=1$, and the Monge-Ampere operator $H_n[u]=\det(u_{i\bar{j}})$ if $k=n$. We also define $\sigma_0=1, \sigma_k=0, \forall k>n$ (see, e.g., [3]).

Definition 1.1. *Let A be an $n\times n$ real symmetric matrix, and denote a symmetric convex cone as*

$$\Gamma_k=\left\{A: \sigma_j(A)>0,\ 1\le j\le k\right\}.$$

Then we say u is k-subharmonic if the complex Hessian $H[u]\in\bar{\Gamma}_k$. We also say that u is plurisubharmonic if $k=n$ and subharmonic if $k=1$.

We introduce some properties about σ_k for later proof (also see, e.g., [3]).

Property 1. Denote $\sigma_k(\lambda|i)$ as taking $\lambda_i=0$ in $\sigma_k(\lambda)$. For $1\le k,i\le n$ and $\lambda\in\mathbb{R}^n$

$$\sigma_k(\lambda)=\sigma_k(\lambda|i)+\lambda_i\sigma_{k-1}(\lambda|i).$$

Property 2. For all $\lambda\in\Gamma_k=\{\lambda\in\mathbf{R}^n: \sigma_j(\lambda)>0,\ 1\le j\le k\}$, with $2\le k\le n$, we have

$$\sigma_{l-1}^2(\lambda)\ge\sigma_l(\lambda)\sigma_{l-2}(\lambda),\quad \forall\, 2\le l\le k.$$

We consider the following Dirichlet problem for $2\le k\le n$:

$$\begin{cases} u \text{ is } k\text{-subharmonic}, & \\ H_k[u]=f, & x\in\Omega, \\ u=\phi, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $f\in C^m(\bar{\Omega})$ is non-negative, $\phi\in C^\infty(\partial\Omega)$, and Ω is Γ_k-pseudoconvex with smooth boundary (k=n i.e. strongly pseudoconvex, see, [4]). The condition u be k-subharmonic is imposed for uniqueness (see, [4,5]). When $k=n$, the corresponding equation is complex Monge-Ampere equation which has been studied by many authors (see, e.g., [6–10]). One of important results is given by Caffarelli et al. [11] which proves that there exists a C^∞ solution to this problem provided that $f\in C^\infty$ is non-vanishing on $\bar{\Omega}$. The result has recently been generalized by Li [4] to the k-Hessian operator (in fact more cases). However, when f is degenerate this is not always true. In this paper we consider what happens in the special case where $f\ge 0$ is radially symmetric.

The problem can be stated as follows. Let B denote the unit ball in $\mathbb{C}^n$. Given $f\ge 0$ on B, find a k-subharmonic function $u\in C^2(B)$ such that

$$\begin{cases} H_k[u]=f(|z|), & z\in B, \\ u=0, & z\in\partial B. \end{cases} \tag{1.2}$$

A radial function u can be considered simply as a function of one real variable r. So in Section 2, we will compute $H_k[u]$ directly, obtaining a non-linear ordinary differential equation $H_k[u](r)=f(r)$. This equation is then solved by two integrations, giving u in terms of f. We have following results