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A Difference Scheme Approximation for Inhomogeneous Schrödinger Flows into S^2

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Abstract. Results concerning existence, uniqueness and local regularity of Inhomogeneous Schrödinger flow from $\mathbb{T}_R^d$ into S^2, are presented by using the difference method, where $\mathbb{T}_R^d = \mathbb{R}^d/(R\cdot\mathbb{Z})^d$.

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1 Introduction

The well-known inhomogeneous Heisenberg spin (IHS) chain system (also called inhomogeneous ferromagnetic spin chain system) is given by

$$\partial_t S(x,t) = \sigma(x,t)\big\{S(x,t)\times\triangle S(x,t)\big\} + S(x,t)\times\big\{\nabla\sigma(x,t)\cdot\nabla S(x,t)\big\},\quad x\in\mathbb{T}_R^d, \tag{1.1a}$$

$$S(x,0) = S_0, \tag{1.1b}$$

where S_0 is the initial data, $S(x,t)\in S^2\subset\mathbb{R}^3$, $\sigma(x,t)$ is a positive real function on $\mathbb{T}_R^d\times[0,\infty)$, $\times$ denotes the cross product in $\mathbb{R}^3$ and $\triangle$ is the Laplace operator on $\mathbb{R}^d$(see [1–5]).

For a smooth map $u:(M,g)\to(N,h)$, we recall that, in local coordinates, the tension field can be written as

$$\tau^\alpha(u) = \triangle u^\alpha + g^{ij}\Gamma^\alpha_{\beta\gamma}(u)\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j},$$

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where $\triangle$ is the Laplace-Beltrami operator on M with respect to the metric g and $\Gamma^{\alpha}_{\beta\gamma}$ are the Christoffel symbols of the target manifold (N,h). Thus, it is easy to see that the IHS chain system can be written as

$$\partial_t S(x,t) = \sigma(x,t) J\big(S(x,t)\big)\Big(\tau(S(x,t))\Big) + J\big(S(x,t)\big)\big(\nabla\sigma(x,t)\cdot\nabla S(x,t)\big), \qquad x\in\mathbb{T}^d_R.$$

Here $J(S(x,t)) \equiv S(x,t)\times : T_{S(x,t)}S^2 \to T_{S(x,t)}S^2$ is just the complex structure on S^2. This shows that the IHS chain system is a nonlinear Schrödinger equation into S^2 with variable coefficients. If $\sigma(x,t)\equiv\sigma(x)$ is a real function of x, it can be viewed as an infinite dimensional Hamiltonian system with respect to the inhomogeneous energy functional (see [6,7]) given by

$$E_\sigma(S) = \int_{\mathbb{T}^d_R} |dS|^2 \sigma(x)\mathrm{d}x,$$

where $|dS|^2$ denotes the Hilbert-Schmidt norm of the tangent map $dS : T\mathbb{T}^d_R \to TS^2$.

Generally, the inhomogeneous Schrödinger flow from a Riemannian manifold M into a symplectic manifold (N,J) can be defined by

$$\partial_t S = \sigma(x) J(S)\tau(S) + J(S)(\nabla\sigma(x)\cdot dS),$$

with initial value

$$S(x,0) = S_0(x) : \mathbb{T}^d_R \to N.$$

Here σ is a positive real function on M. Especially, when M is compact and $\dim(M)\neq 2$, the above inhomogeneous flow can be transformed into a Schrödinger flow by a conformal transformation of the metric on the domain manifold.

For the homogeneous case (i.e. $\sigma\equiv 1$), Sulem *et al.* in [8] proved that the local existence of the solutions to the above initial value problem (1.1) and the global existence with small initial value. Zhou *et al.* [9] showed that for smooth initial data there exists a unique smooth solution for the Cauchy problem of the ferromagnetic spin chain system from S^1 into S^2. In [10,11], Wang proved the existence of a global weak solution for the Cauchy problem of the ferromagnetic spin system from any closed manifold into S^2 with or without external magnetic field (see also [12]).

In [13], Ding and Wang studied the Schrödinger flow for maps from a compact Riemannian manifold into a symplectic manifold. For a symplectic manifold (N,J) with symplectic form ω, where J is an almost complex structure on N such that $h(\cdot,\cdot) = \omega(\cdot, J\cdot)$ is a Riemannian metric, the Schrödinger flow for maps from (M,g) into (N,h) is defined by the equation $\partial_t S = J(S)\tau(S)$. It can be viewed as an infinite dimensional Hamiltonian system. When M is the unit circle and (N,J) is a Kähler manifold, Ding and Wang [13] proved that the Schrödinger flow admits a unique local smooth solution. Furthermore, when (N,J) is a compact Riemann surface with constant curvature, they showed that the solution exists globally by exploiting a conservative law. Later, they also showed that the