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On Global Smooth Solution of A Semi-Linear System of Wave Equations in $\mathbb{R}^3$

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Abstract. In this paper we consider the Cauchy problem for a semi-linear system of wave equations with Hamilton structure. We prove the existence of global smooth solution of the system for subcritical case by using conservation of energy and Strichartz's estimate. On the basis of Morawetz-Pohožev identity, we obtain the same result for the critical case.

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1 Introduction and main results

This paper is concerned with the Cauchy problem for the non-linear system of wave equations with Hamilton structure in $\mathbb{R}_+^{3+1}$

$$\begin{cases} u_{tt}-\Delta u=-F_1(|u|^2,|v|^2)u, \\ v_{tt}-\Delta v=-F_2(|u|^2,|v|^2)v, \\ u(0)=\varphi_1(x), \quad u_t(0)=\psi_1(x), \\ v(0)=\varphi_2(x), \quad v_t(0)=\psi_2(x), \end{cases} \tag{1.1}$$

where there exists a function $F(\lambda,\mu)$ such that

$$\frac{\partial F(\lambda,\mu)}{\partial \lambda}=F_1(\lambda,\mu), \quad \frac{\partial F(\lambda,\mu)}{\partial \mu}=F_2(\lambda,\mu). \tag{1.2}$$

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The linear case $F_j = m_j$, where $m_j \in \mathbb{R}$ for $j = 1,2$, corresponds to the classical Klein-Gordon system in relativistic particle physics. The constants m_j may be interpreted as masses and hence are generally assumed to be nonnegative. In order to model also non-linear phenomenon like quantization, in the 1950s systems of type (1.1) with non-linearities like $F_j = m_j + f_j$ were proposed as models in relativistic quantum mechanics with local interaction, see, e.g., [1, 2].

Various other models involving non-linearities F_j depending also on $u_t, v_t, \nabla u$ and ∇v have been studied [3]. To limit our paper to a reasonable length, we restrict our study to non-linearities depending only on u,v, i.e., the semi-linear case.

Without loss of generality, and since all important features of our problem already seem to exist in this case, we confine ourselves to real-valued solutions of (1.1). Moreover, we need to impose the following assumptions on the semi-linearities to ensure that (1.1) always has a global solution.

(H1)

$$|F_1| + |\lambda F_{11}| + |\lambda^{\frac{1}{2}}\mu^{\frac{1}{2}}F_{12}| + |F_2| + |\lambda^{\frac{1}{2}}\mu^{\frac{1}{2}}F_{21}| + |\mu F_{22}| \le C\big(1 + \lambda^{\frac{k-1}{2}} + \mu^{\frac{k-1}{2}}\big), \tag{1.3}$$

where $F_{i1} = \partial F_i/\partial\lambda,\ \ F_{i2} = \partial F_i/\partial\mu,\ \ i = 1,2.$

(H2)

$$F(\lambda,\mu) \ge 0, \quad F(0,0) = 0, \quad \lambda^{\frac{k+1}{2}} + \mu^{\frac{k+1}{2}} \le C_0[1 + \frac{1}{2}F(\lambda,\mu)]. \tag{1.4}$$

(H3)

$$\lambda F_1(\lambda,\mu) + \mu F_2(\lambda,\mu) \ge 2F(\lambda,\mu), \quad k = 5. \tag{1.5}$$

(H4)

$$F(\lambda,\mu) \le C(1 + \lambda^{\frac{k+1}{2}} + \mu^{\frac{k+1}{2}}). \tag{1.6}$$

It is easy to verify that

$$F_1(\lambda,\mu) = \lambda^2 + \mu, \quad F_2(\lambda,\mu) = \mu^2 + \lambda, \quad F(\lambda,\mu) = \frac{1}{3}\lambda^3 + \frac{1}{3}\mu^3 + \lambda\mu$$

satisfy (H1)-(H4) with $k = 5$.

It is known that the energy associated with (1.1) is defined by

$$\begin{aligned} E(u,v;t) = &\frac{1}{2}\int_{\mathbb{R}^3} \big[|u_t(x,t)|^2 + |v_t(x,t)|^2 + |\nabla u(x,t)|^2 \\ &+ |\nabla v(x,t)|^2 + F(|u(x,t)|^2, |v(x,t)|^2)\big]\,\mathrm{d}x \\ \triangleq &\frac{1}{2}\int_{\mathbb{R}^3} \big[|u'(x,t)|^2 + |v'(x,t)|^2 + F(|u(x,t)|^2, |v(x,t)|^2)\big]\,\mathrm{d}x. \end{aligned} \tag{1.7}$$

Notice that the above energy involves two kinds of terms: the kinetic term and the potential term involving semi-linearity $F(|u|^2, |v|^2)$. To make sure that the potential energy