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# A CAHN-HILLIARD TYPE EQUATION WITH GRADIENT DEPENDENT POTENTIAL*

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Dedicated to the 70th birthday of Professor Li Tatsien

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**Abstract** We investigate a Cahn-Hilliard type equation with gradient dependent potential. After establishing the existence and uniqueness, we pay our attention mainly to the regularity of weak solutions by means of the energy estimates and the theory of Campanato Spaces.

**Key Words** Cahn-Hilliard equation; existence; uniqueness; regularity.

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## 1. Introduction

In this paper, we consider the initial boundary value problem for the following Cahn-Hilliard type equation with gradient dependent potential

$$\frac{\partial u}{\partial t} + \operatorname{div}\left(K\nabla\Delta u - \overrightarrow{\Phi}(\nabla u)\right) = 0, \qquad (x,t)\in Q_T, \tag{1.1}$$

$$\nabla u\cdot\nu\Big|_{\partial\Omega} = \mu\cdot\nu\Big|_{\partial\Omega} = 0, \qquad t\in[0,T], \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x\in\Omega, \tag{1.3}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $Q_T = \Omega\times(0,T)$, $\nu$ denotes the unit exterior normal to the boundary $\partial\Omega$, $\mu = K\nabla\Delta u - \overrightarrow{\Phi}(\nabla u)$ is the flux, $K$ is the positive diffusion coefficient, and $\overrightarrow{\Phi} = (\Phi_1,\Phi_2,\cdots,\Phi_N)$ is a smooth vector function from $\mathbb{R}^N$ to $\mathbb{R}^N$.

The problem (1.1)–(1.3) models many interesting phenomena in mathematical biology, fluid mechanics, phase transition, etc. Recently, such type of equations, especially in the case of one spatial dimension have arisen interests to many mathematicians.

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For example, Myers [1] considered the following one dimensional fourth-order diffusion equation,

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left( C\frac{u^3}{3}\frac{\partial^3 u}{\partial x^3} + f(u, u_x, u_{xx}) \right) = 0,$$

which has been proposed to describe the surface tension phenomena in some particular case of thin films. We refer the readers to [2–6] for more examples of one-dimensional models. However, many models in multi-dimensional case may occur from practical problems, see for example [7,8]. Here, we briefly introduce the derivation of the equation (1.1) based on the continuum model for epitaxial thin film growth from King, Stein and Winkler [9]. Let $u(x,t)$ be the height of the film at point $x$ and time $t$. Then $u$ satisfies the following basic equation

$$\frac{\partial u}{\partial t} = g - \nabla \cdot \vec{j} + \eta, \tag{1.4}$$

where $g = g(x,t)$ denotes the deposition flux, $\vec{j} = \vec{j}\,(x,t)$ comprises all processes which move atoms along the surface, $\eta = \eta(x,t)$ is some Gaussian noise. The pivotal step in the phenomenological approach is to expand $\vec{j}$ in $\nabla u$ and powers thereof keeping only "sensible" terms (see [10]). Then, we have

$$\vec{j} = A_1\nabla u + A_2\nabla\Delta u + A_3|\nabla u|^2\nabla u + A_4\nabla|\nabla u|^2, \tag{1.5}$$

where $A_1, A_2, A_3$ and $A_4$ are constants. It can be informed from the work of Ortiz [11] that $A_4 = 0$ if Onsager's reciprocity relations hold. Then, after neglecting the effect of the noise term $\eta$ and the deposition flux $g$, the equation (1.4) reads

$$\frac{\partial u}{\partial t} + A_2\Delta^2 u + A_1\nabla\cdot\left(\frac{A_3}{A_1}|\nabla u|^2 + 1\right)\nabla u = 0. \tag{1.6}$$

It can be informed from [11] that the case of $A_1 > 0, A_3 < 0$ is significative and interesting. After relabeling the constants, we have the following fourth-order diffusion equation

$$\frac{\partial u}{\partial t} + \alpha\Delta^2 u - \beta\nabla\cdot\left(|\nabla u|^2\nabla u\right) + \gamma\Delta u = 0,$$

where $\alpha, \beta$ and $\gamma$ are positive constants. Moreover, from a mathematical point of view it is more satisfactory to generalize the term involving second-order diffusion, and then we have

$$\frac{\partial u}{\partial t} + \alpha\Delta^2 u - \beta\nabla\cdot\vec{\Phi}\,(\nabla u) = 0, \tag{1.7}$$

where $\vec{\Phi}$ is a smooth vector with $\vec{\Phi}(0) = 0$. In fact, a lot of references (for example [12–15] etc) show that the diffusion coefficient is usually dependent on the concentration in the models governed by the Cahn-Hilliard equations or Thin-Film equations. Then, the equality (1.5) should be rewritten as

$$\vec{j} = m(u)\left(A_1\nabla u + A_2\nabla\Delta u + A_3|\nabla u|^2\nabla u + A_4\nabla|\nabla u|^2\right),$$