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# ASYMPTOTIC DECAY TOWARD RAREFACTION WAVE FOR A HYPERBOLIC-ELLIPTIC COUPLED SYSTEM ON HALF SPACE*

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**Abstract** We consider the asymptotic behavior of solutions to a model of hyperbolic-elliptic coupled system on the half-line $\mathbb{R}_+ = (0, \infty)$,

$$u_t + uu_x + q_x = 0, \quad -q_{xx} + q + u_x = 0,$$

with the Dirichlet boundary condition $u(0, t) = 0$. S. Kawashima and Y. Tanaka [Kyushu J. Math., **58**(2004), 211-250] have shown that the solution to the corresponding Cauchy problem behaviors like rarefaction waves and obtained its convergence rate when $u_- < u_+$. Our main concern in this paper is the boundary effect. In the case of null-Dirichlet boundary condition on $u$, asymptotic behavior of the solution $(u, q)$ is proved to be rarefaction wave as $t$ tends to infinity. Its convergence rate is also obtained by the standard $L^2$-energy method and $L^1$-estimate. It decays much lower than that of the corresponding Cauchy problem.

**Key Words** Hyperbolic-elliptic coupled system; rarefaction wave; asymptotic decay rate; half space; $L^2$-energy method; $L^1$-estimate.

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## 1. Introduction

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In this paper, we will consider the initial-boundary value problem on the half-line $\mathbb{R}_+ = (0, \infty)$ for a hyperbolic-elliptic coupled system in radiation hydrodynamics

$$\begin{cases} u_t + uu_x + q_x = 0, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ -q_{xx} + q + u_x = 0, & (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+, \end{cases} \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x) = \begin{cases} = 0, & x = 0, \\ \to u_+, & x \to \infty, \end{cases} \tag{1.2}$$

and boundary condition

$$u(0,t) = 0, \quad t \in \mathbb{R}_+, \tag{1.3}$$

where $u_+ << 1$ is given sufficiently small positive constant.

The first equation is a hyperbolic conservation law and the second is an elliptic equation. Such a hyperbolic-elliptic coupled system appears typically in radiation hydrodynamics, cf. [1]. The system (1.1) is derived as the third-order approximation of the full system describing the motion of radiating gas in thermo-nonequilibrium, while the second-order approximation gives the viscous Burgers equation $u_t + f(u)_x = u_{xx}$, and the first-order approximation gives the inviscid Burgers equation $u_t + f(u)_x = 0$ (see [2]). Hamer [3] studies these equations in the physical respect, especially for the steady progressive shock wave solutions. It is also studied from the mathematical point of view.

In the case of a whole space, Kawashima and Nishibata prove the stability of the traveling wave when $u_- > u_+$, cf. [2, 4, 5]. Kawashima and Tanaka show the stability of the rarefaction wave and the asymptotic rate when $u_- < u_+$. In their proof the second-order approximation is based on the viscous Burgers equation, cf. [6]. For the general flux $f(u)$, Ruan and Zhang show the stability of the rarefaction wave when $u_- < u_+$. While in their proof the second-order approximation is based on the inviscid Burgers equation, cf. [7]. Recently, for the Cauchy problem of a model system of the radiating gas in two space dimensions, Gao and Zhu [8] investigate asymptotic decay toward the planar rarefaction waves. On the other hand, there are a lot of related works concerning the stability of rarefaction waves and viscous shock waves for viscous conservation laws and other system, we refer to [9–17].

But for the half space, the asymptotic states of the solutions is studied until recently. In particular, to our knowledge, no results have been obtained on the asymptotic states of the solutions of (1.1). Liu, Matsumura and Nishihara [18] investigated the asymptotic behaviors of solutions of the initial-boundary value problem to the generalized Burgers equation $u_t + f(u)_x = u_{xx}$ on the half-line. The corresponding convergence rate for the rarefaction wave was obtained by Nakamura [19]. Also, Yang, Zhao and Zhu [20] studied the asymptotic behavior of solutions to a hyperbolic system with relaxation and boundary effect. Motivated by the above investigations, in this paper, we will prove