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ORBITAL INSTABILITY OF STANDING WAVES FOR THE COUPLED NONLINEAR KLEIN-GORDON EQUATIONS*

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Abstract This paper deals with a type of standing waves for the coupled nonlinear Klein-Gordon equations in three space dimensions. First we construct a suitable constrained variational problem and obtain the existence of the standing waves with ground state by using variational argument. Then we prove the orbital instability of the standing waves by defining invariant sets and applying some priori estimates.
Key Words Orbital instability; standing wave; coupled Klein-Gordon equations; invariant set.
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1. Introduction

Recently, some coupled nonlinear wave equations are being investigated extensively (see Reference [1-7]). In his paper[8], Zhang studied the coupled nonlinear Klein-Gordon equations

$$\begin{cases} \phi_{tt} - \Delta\phi + m_1\phi = (a_{11}|\phi|^2 + a_{12}|\psi|^2)\phi, t > 0, x \in R^N, \\ \psi_{tt} - \Delta\psi + m_2\psi = (a_{21}|\phi|^2 + a_{22}|\psi|^2)\psi, t > 0, x \in R^N, \end{cases} \tag{1.1}$$

and obtained the instability of a type of standing waves for (1.1) by proving blow up of solution, where (ϕ, ψ) is a pair of complex-valued functions of $(t, x) \in R^+ \times R^N$, m_j

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and $a_{jk} > 0$ $(j, k = 1, 2)$ are real parameters. In the present paper, we consider the orbital instability of another type of standing waves for (1.1) which is quite different from that in [8] and is obtained without blow up of solution.

(1.1) can be regarded as a model to describe the interaction of two fields with the mass m_1 and the mass m_2,respectively. In this paper, for (1.1), we consider only the case of $N = 3$ and define different functionals and manifold, thus we obtain the existence of the ground state and the orbital instability of the ground state. The method we use for obtaining the ground state is that of Clayton Keller who studied the flow of the equation $u_{tt} + u_t - \Delta u + f(u) = 0$ near a stationary state. Keller showed that the stationary state is a saddle point with an infinite dimensional stable manifold and a nonempty finite dimensional unstable manifold. In [8], Zhang obtained the instability of a type of standing waves for (1.1) by proving blow up in finite time. However, for (1.1), some solutions do not blow up in finite time but exist globally in time. Therefore to solve this problem we are forced to introduce a new estimate that shows the instability of the ground state without blow up for (1.1).

Throughout this paper, it will be assumed that the scalar matrix in (1.1) satisfies

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ is positive definite} \tag{1.2}$$

and $m_j, a_{jk} > 0(j, k = 1, 2)$. In addition, for simplicity, we denote $\int_{R^3} \cdot \mathrm{d}x$ by $\int \cdot \mathrm{d}x$.

2. Existence of the Standing Waves

If a pair of real functions $(u, v) = (u(x), v(x)), x \in R^3$ verifies the semilinear elliptic equations

$$\begin{cases} -\Delta u + m_1 u = (a_{11}u^2 + a_{12}v^2)u \\ -\Delta v + m_2 v = (a_{21}u^2 + a_{22}v^2)v \end{cases} \tag{2.1}$$

and $(u, v) \in H_r^1(R^3) \times H_r^1(R^3) \setminus \{(0, 0)\}$, then $\phi(t, x) = u(x), \psi(t, x) = v(x), t \geq 0, x \in R^3$ verify (1.1), which are the standing wave solutions of (1.1). Here

$$H_r^1(R^3) = \Big\{u, \text{radially symmetric functions on } R^3, \\ \|u\|_{H^1(R^3)} = \Big(\int |\nabla u(x)|^2 \mathrm{d}x + \int |u(x)|^2 \mathrm{d}x\Big)^{\frac{1}{2}} < \infty\Big\}.$$

In the following, the existence of a nontrivial solution to (2.1) is shown. First the action of the solution (u, v) of (2.1) is defined as follows:

$$\begin{aligned} J(u, v) :=& \frac{1}{2}\int (p|\nabla u|^2 + |\nabla v|^2 + pm_1u^2 + m_2v^2)\mathrm{d}x \\ &- \frac{1}{4}\int (pa_{11}u^4 + a_{22}v^4 + 2a_{21}u^2v^2)\mathrm{d}x, \end{aligned} \tag{2.2}$$