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GLOBAL SOLUTIONS TO AN INITIAL BOUNDARY VALUE PROBLEM FOR THE MULLINS EQUATION

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Abstract In this article we study the global existence of solutions to an initial boundary value problem for the Mullins equation which describes the groove development at the grain boundaries of a heated polycrystal, both the Dirichlet and the Neumann boundary conditions are considered. For the classical solution we also investigate the large time behavior, it is proved that the solution converges to a constant, in the $L^\infty(\Omega)-$norm, as time tends to infinity.

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1. Introduction

In the present article we are interested in the global existence of solutions to an initial boundary value problem for the Mullins equation which describes the groove development at the grain boundaries of a heated polycrystal. When the weak solution happens to be classical we also investigate the large time behavior of the solution. This model was proposed by Mullins in 1957, see [1]. In the classical theory of thermal grooving, two principal mechanisms for mass transport on a metal surface can be distinguished, the evaporation-condensation and the surface diffusion. For some metals like magnesium, the first mechanism plays a dominated role after a very short time. While for some other metals, such as gold, the second mechanism dominates the process for a very long time. We refer to [1] for more details. The initial boundary value problem reads

$$u_t = D\left(1+u_x^2\right)^{-1} u_{xx}, \tag{1.1}$$

$$u|_{\partial\Omega} = 0, \tag{1.2}$$

$$u|_{t=0} = u_0, \tag{1.3}$$

where the equation (1.1) must be satisfied in $Q_T = (0,T) \times \Omega$, T is a certain real number, $\Omega = (a,b)$, and $\partial\Omega = \{a,b\}$ with a,b being real numbers such that $a < b$. We also consider the Neumann boundary condition, namely (1.2) is replaced by

$$u_x|_{\partial\Omega} = 0. \tag{1.4}$$

Here $u = u(t,x)$ is the unknown, u_0 is the initial data which is given, where x, u are Cartesian coordinates and t is the time. D is a constant defined by

$$D = \frac{p_0 \gamma \omega^2}{(2\pi M)^{\frac{1}{2}} (kT)^{\frac{3}{2}}},$$

where p_0 is the vapor pressure in equilibrium with a plane surface (the curvature $K = 0$), γ is the surface-free energy per unit area, ω is the molecular volume, M is the weight of a molecule, and k is the Boltzmann constant and T is the absolute temperature, respectively. For simplicity we assume that $D = 1$. As we shall see later on, we state the existence theorem of solutions to the problems for both the Dirichlet and the Neumann boundary conditions, we investigate mainly the problem with Dirichlet boundary condition since many parts of the proofs for the two problems are similar, however, we still state the key ingredients in the proof of the theorem for the problem with the Neumann boundary condition, which is crucially different from those arising in the Dirichlet problem.

Equation (1.1) is a model for thermal grooving of the first mechanism. We choose the free energy function as

$$f(u_x) = \frac{\nu}{2}|u_x|^2,$$

suppose that u is a classical solution to (1.1) – (1.3), then one has

$$\begin{aligned} \frac{d}{dt}\int_\Omega f(u_x(t,x))dx &= \nu \int_\Omega u_x u_{xt} = -\nu \int_\Omega u_{xx} u_t dx \\ &= -\nu \int_\Omega \left(1 + u_x^2\right)^{-1} u_{xx}^2 dx \\ &\le 0. \end{aligned} \tag{1.5}$$

Therefore, the second law of thermodynamics is satisfied. If we define

$$J = \int^{u_x} \frac{dy}{1+y^2},$$

we find that (1.1) become $u_t = J_x$, so J is a flux, (1.1) defines a gradient flow.

On the other hand, we can easily see that (1.1) is non-uniformly parabolic since the coefficient of its leading term may decay to zero as u_x tends to infinity. Thus, we modify the equation to a uniformly parabolic one, to solve this approximate problem we employ an existence theorem for quasilinear parabolic equations, see e.g. Ladyzenskaya,