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THE HAMILTONIAN STRUCTURE OF TWO INTEGRABLE EXPANDING MODELS

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Abstract In this paper, an extended loop algebra is constructed from which an isospectral problem established. It follows that the integrable couplings of the Tu hierarchy and M-AKNS-KN hierarchy are obtained, and their Hamilton structures are presented by the quadratic-form identity. Moreover, we guarantee that the expanding model we obtained are also Liouville integrable.

Key Words Integrable coupling; Hamiltonian structure; quadratic-form identity.
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1. Introduction

Guizhang Tu proposed an efficient approach for searching for integrable Hamiltonian hierarchy, which was usually called Tu scheme [1]. By use of Tu scheme, a lot of integrable systems with physics significance have been obtained [2-6]. But only the spectral problem of the matrix form is made use of for constructing the Hamiltonian structure of integrable systems to some solitary equations which can be obtained by vector loop algebra and multi-component matrix loop algebra, the Hamiltonian structures can't be derived out by the trace identity. Recently, a new method is proposed to solve the problem by use of quadratic-form identity [7]. In this paper, we firstly deduce the couplings of Tu hierarchy and M-AKNS-KN hierarchy, by extending spectral problem, and then obtain the corresponding Hamiltonian structure by use of a quadratic-form identity, in the end, we can conclude that the two couplings are Liouville integrable.

Assume that G is an s-dimensional Lie algebra with the basis $\{e_1, e_2, \cdots, e_s\}$, and let $a = \sum_{k=1}^{s} a_k e_k, b = \sum_{k=1}^{s} b_k e_k, a, b \in G$, the resulting loop algebra $\tilde{G}$ is with the basis $e_k(m) = e_k\lambda^m (1 \le k \le s), [e_k(m), e_j(n)] = [e_k, e_j]\lambda^{m+n}$.

The vector form of G is given by

$$\tilde{G} = \{a = (a_1, a_2, \cdots, a_s)^T, a_k = \sum_{m=1}^{s} a_{km}\lambda^m, [a, b] = c = (c_1, c_2, \cdots, c_s)^T\}. \quad (1)$$

The linear isospectral problem established by $\tilde{G}$ is as follows

$$\begin{cases} \psi_\partial = [U, \psi], U, V, \psi \in \tilde{G}, \lambda_t = 0, \\ \psi_t = [V, \psi], \partial = \sum_{k=1}^{n} \alpha_k \dfrac{\partial}{\partial x_k}, \quad \alpha_k \text{ are constants.} \end{cases} \tag{2}$$

The compatibility $\psi_{\partial t} = \psi_{t\partial}$ leads to the zero-curvature equation

$$U_t - V_\partial + [U, V] = 0, \tag{3}$$

which results in stationary zero-curvature equation

$$V_\partial = [U, V]. \tag{4}$$

For λ and u_i $(i = 1, 2, \cdots, p)$ in $U = U(\lambda, u) = \sum_{i=1}^{s} U_i e_i$, we define the rank numbers denoted by $\mathrm{rank}(\lambda)$ and $\mathrm{rank}(u_i)$ so that $\mathrm{rank}(U_i e_i) = \alpha = \mathrm{const}, 1 \leq i \leq s$, and simultaneously we call U the same rank, or homogeneous in rank, denoted by

$$\mathrm{rank}(U) = \mathrm{rank}(\partial) = \mathrm{rank}(\frac{\partial}{\partial x_i}) = \alpha, i = 1, 2, \cdots, n. \tag{5}$$

If V_1 and V_2 are the same-rank solutions to (4) and satisfy $V_1 = \gamma V_2$, let $[a, b]^T = a^T R(b), a, b \in \tilde{G}$, the constant matrix $F = (f_{ij})_{s\times s}$ satisfies

$$F = F^T, \qquad R(b)F = -(R(b)F)^T. \tag{6}$$

Let us define a quadratic-form functional

$$\{a, b\} = a^T F b, \quad \forall a, b \in \tilde{G}, \tag{7}$$

then we obtain

$$\frac{\delta}{\delta u_i}\{V, U_\lambda\} = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\gamma \{V, \frac{\partial U}{\partial u_i}\}, i = 1, 2, \cdots, p, \tag{8}$$

where γ is a constant to be determined, we call (8) the quadratic-form identity.

2. The Hamiltonian Structure of an Integrable Coupling in Vector Form

2.1 The integrable coupling of Tu hierarchy

Let $G_6 = \{a = (a_1, a_2, \cdots, a_6)^T\}$ be a 6-dimensional linear space with the commuting relations

$$[a, b] = \begin{pmatrix} a_3 b_2 - a_2 b_3 \\ a_1 b_3 - a_3 b_1 \\ a_1 b_2 - a_2 b_1 \\ a_3 b_5 - a_5 b_3 + a_6 b_2 - a_2 b_6 \\ a_1 b_6 - a_6 b_1 + a_4 b_3 - a_3 b_4 \\ a_1 b_5 - a_5 b_1 + a_4 b_2 - a_2 b_4 \end{pmatrix}. \tag{9}$$