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THE CAUCHY PROBLEM OF THE MODIFIED KAWAHARA EQUATION

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Abstract In this paper, we consider the local and global solutions for the modified Kawahara equation with data in the homogeneous and nonhomogeneous Besov space and the scattering result for small data. The techniques to be used are adapted from Kato's smoothing effect and the maximal function (in time) estimate for the free Kawahara operator $e^{-\gamma t\partial_x^5}$.

Key Words Modified Kawahara equation; Cauchy problem; Littlewood-Paley decomposition; Besov space.
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1. Introduction

In this paper, we consider the Cauchy problem of the modified Kawahara equation

$$\begin{cases} \partial_t u + \gamma\partial_x^5 u = -\,\partial_x\big(u^k\big), \quad x \in \mathbb{R}, \quad t \in \mathbb{R}; \\ u(0) = \varphi(x). \end{cases} \tag{1.1}$$

where $\gamma \neq 0$ is real constant, $\varphi \in \dot{\mathcal{B}}^{s_k}_{2,q}$ with $s_k = \frac{1}{2} - \frac{4}{k-1}$, which is is critical by scaling..

The modified Kawahara equation such as (1.1) has been intensively studied. It is first proposed by Kawahara [1] as a model describing capillary-gravity waves for the Bond number near one third and the Froude number near 1. There are some similarities and differences between (1.1) and Kdv equation. They all have the solitary wave solutions, but the solitary wave solutions of the Kawahara equation have oscillatory trails, the solitary wave solutions of the Kdv equation are non-oscillatory. (see [2-10])

The local well-posedness of the Cauchy problem (1.1) is generally obtained by using a fixed point argument on the suitable resolution space for its corresponding integral formulation

$$u = \mathcal{G}u = \mathcal{W}(t)\varphi - \int_0^t \mathcal{W}(t-s)\partial_x(u^k)(s)ds, \tag{1.2}$$

where $\mathcal{W}(t)=e^{-\gamma t\partial_x^5}$ is the unitary group. It is well known that (1.1) is global well-posedness in the Sobolev space $\mathcal{H}^s(\mathbb{R}), s\geq 2$, for $k=1,...,7$. This is a direct consequence of the local well-posedness in $\mathcal{H}^2(\mathbb{R})$ with the conservation of the mass and energy quantities.

$$M(u)=\int_{-\infty}^{\infty}u^2(t,x)dx,$$

and

$$E(u)=\int_{-\infty}^{\infty}\frac{1}{2}\big(\partial_x^2 u\big)^2-\frac{u^{k+1}}{k+1}\,dx. \tag{1.3}$$

When $k\geq 8$, we can obtain the global wellposedness in $\mathcal{H}^2(\mathbb{R})$ under the smallness assumptions on the initial data.

The general Cauchy problem

$$\begin{cases}\partial_t u+\beta\partial_x^3 u+\gamma\partial_x^5 u=-u\partial_x u, & x\in\mathbb{R},\quad t\in\mathbb{R};\\ u(0)=\varphi(x).\end{cases} \tag{1.4}$$

is recently studied in [11-13]. By the Strichartz type estimates, it is well known that (1.4) is local well-posedness if $\varphi\in\mathcal{H}^r(\mathbb{R})$ with $r>-1$. Combining with the energy conservation law, they obtain that it is global well-posedness if $\varphi\in\mathcal{L}^2(\mathbb{R})$. In particular, these results are improved recently in [13] by improving the bilinear estimates and performing the almost conservation law [14].

We mainly study the Cauchy problem (1.1) with $k\geq 5$ by Kato's smoothing effect and the maximal function (in time) estimate for the free Kawahara operator in the homogeneous and nonhomogeneous Besov space, which contains the self-similar structure. Note that the Cauchy problem (1.4) has not the self-similar solutions.

The paper is organized as follows.

In Section 2, we give the linear estimates for the phase localized functions by Kato smooth effect and the maximal function (in time) estimate. In Section 3, we obtain the estimates for the linear term and nonlinear term. In Section 4, we give the main theorems and their corresponding proof. Section 5 is devoted to prove the scattering result for small data.

Last, we give some notations:

For s a real number, s^+ denotes a number slightly larger than s; p and p' are Hölder dual exponents. We will use the Lebesgue spaces $\mathcal{L}_x^p\mathcal{L}_t^q$ and $\mathcal{L}_t^q\mathcal{L}_x^p$ respectively equipped with the norms

$$\|f(t,x)\|_{\mathcal{L}_x^p\mathcal{L}_t^q}=\big\|\|f(t,x)\|_{\mathcal{L}_t^q}\big\|_{\mathcal{L}_x^p}\quad\text{and}\quad\|f(t,x)\|_{\mathcal{L}_t^q\mathcal{L}_x^p}=\big\|\|f(t,x)\|_{\mathcal{L}_x^p}\big\|_{\mathcal{L}_t^q}.$$

Sometimes we will need the local in time version of those spaces. We denote them by $\mathcal{L}_x^p\mathcal{L}_T^q$ and $\mathcal{L}_T^q\mathcal{L}_x^p$, which are equipped with the norms

$$\|f(t,x)\|_{\mathcal{L}_x^p\mathcal{L}_T^q}=\big\|\|f(t,x)\|_{\mathcal{L}_{[-T,T]}^q}\big\|_{\mathcal{L}_x^p}\quad\text{and}\quad\|f(t,x)\|_{\mathcal{L}_T^q\mathcal{L}_x^p}=\big\|\|f(t,x)\|_{\mathcal{L}_x^p}\big\|_{\mathcal{L}_{[-T,T]}^q}.$$