J. Partial Diff. Eqs. **19**(2006), 341–358
©International Academic Publishers Vol.19 No.4

A CARLEMAN ESTIMATE ON GROUPS OF HEISENBERG TYPE*

Han Junqiang and Niu Pengcheng
(Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, 710072, China)
(E-mail: southhan@163.com)
(Received May. 24, 2005; revised Sup. 11, 2006)

Abstract A Pohozaev-Rellich type identity for the p-sub-Laplacian on groups of Heisenberg type, G, is given. A Carleman estimate for the sub-Laplacian on G is established and, as a consequence, a unique continuation result is proved.
Key Words Pohozaev-Rellich type identity, Carleman estimate, unique continuation, sub-Laplacian, group of Heisenberg type.
2000 MR Subject Classification 35H99, 43A80.
Chinese Library Classification O175.2.

1. Introduction

Let G be a Carnot group of step r, with Lie algebra $\mathrm{g} = \oplus_{j=1}^{r} V_j$. Suppose that g is equipped with a scalar product with respect to which the V_j's are mutually orthogonal. We use the exponential map: $\exp : \mathrm{g} \to G$ to define analytic maps: $\xi_i : G \to V_i, i = 1, \dots, r$, through the equation $g = \exp(\xi_1(g) + \dots + \xi_r(g))$. Here, $\xi(g) = \xi_1(g) + \cdots + \xi_r(g)$ is such that $g = \exp(\xi(g))$. With $m = \dim(V_1)$, the coordinates of the projection ξ_1 in the basis $X_1, \dots, X_m$ is denoted by $x_1 = x_1(g), \dots, x_m = x_m(g)$ and we set $x = x(g) = (x_1(g), \dots, x_m(g)) \in \mathbb{R}^m$. Fix an orthogonal basis $Y_1, \dots, Y_k$ of V_2 and define the exponential coordinates in the second layer V_2 of a point $g \in G$ by $y = (y_1, \dots, y_k) \in \mathbb{R}^k$.

For a Carnot group of step two with Lie algebra $\mathrm{g} = V_1 \oplus V_2$, the map $J : V_2 \to End(V_1)$ defined by

$$< J(\xi_2)\xi_1^{'}, \xi_1^{''} >=< \xi_2, [\xi_1^{'}, \xi_1^{''}] >, \quad \text{for } \xi_2 \in V_2 \text{ and } \xi_1^{'}, \xi_1^{''} \in V_1.$$

A Carnot group of step two, G, is called of Heisenberg type, if for every vector $\xi_2 \in V_2$ with $|\xi_2| = 1$, the map $J(\xi_2) : V_1 \to V_1$ is orthogonal. (see [1]).

*The project supported by National Natural Science Foundation of China, Grant No. 10371099.

The sub-Laplacian on the group of Heisenberg type G is given by

$$L=\sum_{j=1}^{m} X_j^2, \tag{1.1}$$

where $\{X_1,\dots,X_m\}$ is the basis of V_1. The p-sub-Laplacian on G is

$$L_p u=\sum_{j=1}^{m} X_j\left(|Xu|^{p-2}X_j u\right), \tag{1.2}$$

for a function u on G.

Kaplan in [1] solved the fundamental solutions for the sub-Laplacian on the group of Heisenberg type G. Recently, Capogna, Danielli and Garofalo [2] obtained the fundamental solutions of p-sub-Laplacian on such groups. Garofalo and Vassilev [3, 4] also studied positive solutions to the CR Yamabe problem on G. Garofalo [5] obtained a Carleman estimate and unique continuation for the generalized Baouendi-Grushin operator. Zhang, Niu and Wang [6] considered the similar properties for the sub-Laplacian on the Heisenberg group.

The purpose of the present paper is to prove a Carleman estimate for L and unique continuations for solutions of the differential inequality (5.1) below.

In Section 2 we collect several basic results on the group of Heisenberg type and the sub-Laplacian. Section 3 is devoted to prove the Pohozaev-Rellich type identity for the p-sub-Laplacian. In Section 4 a Carleman estimate for the sub-Laplacian is established under suitable assumptions. As a consequence of it, the last section contains a unique continuation result.

2. Some Known Facts

Suppose that G is a group of Heisenberg type. As stated in [3, 4], it has

$$X_j=\frac{\partial}{\partial x_j}+\frac{1}{2}\sum_{i=1}^{k} <[\xi,X_j],Y_i>\frac{\partial}{\partial y_i}, j=1,\dots,m, \tag{2.1}$$

where $\xi=\xi_1+\xi_2\in \mathrm{g}=V_1\bigoplus V_2$, $x=(x_1,\dots,x_m)\in\mathbb{R}^m$, $y=(y_1,\dots,y_k)\in\mathbb{R}^k$.

For a function u on G, we denote the horizontal gradient by $Xu=(X_1u,\dots,X_mu)$ and let $|Xu|=\left(\sum_{j=1}^m |X_ju|^2\right)^{\frac{1}{2}}$. A family of non-isotropic dilation on G is

$$\delta_\lambda(x,y)=(\lambda x,\lambda^2 y),\qquad \lambda>0,(x,y)\in G. \tag{2.2}$$

A homogeneous dimension of G is $Q=m+2k$. The generator of the group $\{\delta_\lambda\}_{\lambda>0}$ is

$$Z=\sum_{j=1}^{m} x_j\frac{\partial}{\partial x_j}+2\sum_{i=1}^{k} y_i\frac{\partial}{\partial y_i}. \tag{2.3}$$