J. Partial Diff. Eqs. **18**(2005), 114–120
©International Academic Publishers Vol.18 No.2

ALGEBRAIC METHODS IN PARTIAL DIFFERENTIAL OPERATORS

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(Received May 31, 2004; revised Feb. 5, 2005)

Abstract In this paper we build a class of partial differential operators L having the following property : if u is a meromorphic function in $\mathbb{C}^n$ and Lu is a rational function $\frac{A}{q}$, with q homogenous, then u is also a rational function.

Key Words Chow's lemma; regular partial differential operators; meromorphic functions; homogeneous polynomials; Euler's differential operator.

2000 MR Subject Classification 35G05, 34A20.

Chinese Library Classification O175.4.

1. Introduction

On $\mathbb{M}(C^n)$, the space of meromorphic functions on $\mathbb{C}^n$, we consider the operator

$$\begin{aligned} L: \quad & \mathbb{M}(C^n) \to \mathbb{M}(C^n) \\ & u \to \sum_{|i|=0}^{m} a_i \partial^i u \end{aligned}$$

in which $\partial^i = \dfrac{\partial^{|i|}}{\partial x_1^{i_1} .... \partial x_n^{i_n}}$, $i = (i_1, ..., i_n), | \, i \, | = i_1 + ... + i_n$, and $a_i \in \mathbb{C}[x_1, ..., x_n]$. We denote by Ω^n the space $\{\dfrac{A}{q} \; / \; A \in \mathbb{C}[x_1, ..., x_n], \, q \in \mathbb{C}[x_1, ..., x_n] \backslash \{0\}$ and q homogeneous$\}$, here the operator L is called regular if it satisfies the following property : if Lu belongs to Ω^n for some u in $\mathbb{M}(C^n)$, then u belongs to Ω^n.

The aim of this article is to blued a class of regular partial differential operators, with any order m. Acting in $\mathbb{M}(C^n)$, we present an algebraic method essentially based on Chow's lemma (see [1]), and some particular properties of the classical Euler's (cited below) differential operator E.

This work is composed of two parts, the first part, more technical, is devoted to prove that $P(E)$ is regular, where P is an arbitrary polynomial in $\mathbb{C}[x] \backslash \{0\}$; in the second part we will try to express $P(E)$ in more explicit form.

Besides the main result it presents, this work may be considered as illustrating, in some particular cases, the use of the algebraic methods to study the partial differential operators.

Notations :

$E = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$ is the classical Euler's differential operator.

$x^i = x_1^{i_1} x_2^{i_2} .... x_n^{i_n}$,

$C_{|i|}^i = \frac{|\, i \,|!}{i_1! i_2! ... i_n!}$.

$\mathbb{P}^n(\mathbb{C})$ is the projective space.

2. The Fundamental Theorem

Theorem 1 *Let $b_0, b_1, b_2 ..., b_m$, be $m+1$ complex numbers not all equal to zero. The differential operator $\sum_{|i|=0}^{m} a_i\ \partial^i$ is regular, provided that $a_i = b_{|i|}\ C_{|i|}^i\ x^i$, for all i, $|\, i \,| \leq m$.*

To prove the fundamental theorem we need the following two lemmas :

Lemma 1 *For any arbitrary polynomial $P \in \mathbb{C}[x] \setminus \{0\}$, the differential operator $P(E)$ is regular.*

Proof To do that, we need some preliminary remarks :

1. we know that $\mathbb{C}$ is algebraically closed then : $P(E) = a(E - r_1)...(E - r_m)$ where r_i are the roots of $P(x)$ counted with their multiplicities and $a \in \mathbb{C}^*$.
2. if each $(E - r_i)$ is regular, then $P(E)$ is regular.
3. we see that : $E(x_1^{i_1} x_2^{i_2} .... x_n^{i_n}) = (i_1 + i_2 + ... + i_n) x_1^{i_1} x_2^{i_2} .... x_n^{i_n}$. $\qquad (*)$
4. let $q \in \mathbb{C}[x_1, ..., x_n] \setminus \{0\}$ homogeneous of degree λ et $u \in \mathbb{M}(C^n)$, from $(*)$ we obtain $E(qu) = \lambda qu + qE(u)$. $\qquad (**)$

Let $r \in \mathbb{C}$, now we will prove that $(E - r)$ is regular. Let

$$\begin{cases} E(u) - ru = \dfrac{A}{q}, \\ u \in \mathbb{M}(C^n) \end{cases}$$

where $A \in \mathbb{C}[x_1, ..., x_n]$ and $q \in \mathbb{C}[x_1, ..., x_n] - \{0\}$ homogeneous of degree λ. With $(**)$, $E(u) - ru = \frac{A}{q}$ implies that $E(qu) - (\lambda + r)qu = A$. Let $v = qu$ and $\alpha = \lambda + r$, then :

$$E(v) - \alpha v = A. \tag{1}$$

We distinguish three cases :