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GLOBAL C^1 SOLUTION TO THE INITIAL-BOUNDARY VALUE PROBLEM FOR DIAGONAL HYPERBOLIC SYSTEMS WITH LINEARLY DEGENERATE CHARACTERISTICS

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Abstract We prove that the C^0 boundedness of solution implies the global existence and uniqueness of C^1 solution to the initial-boundary value problem for linearly degenerate quasilinear hyperbolic systems of diagonal form with nonlinear boundary conditions. Thus, if the C^1 solution to the initial-boundary value problem blows up in a finite time, then the solution itself must tend to the infinity at the starting point of singularity.

Key Words Initial-boundary value problem, global C^1 solution, quasilinear hyperbolic system of diagonal form, linearly degenerate characteristics.

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1. Introduction

For first order quasilinear hyperbolic systems, generically speaking, the classical solution exists only locally in time and the singularity will appear in a finite time (see [1-3] and the references therein). In some cases, however, the global existence of classical solution can be obtained. For example, for the Cauchy problem for quasilinear hyperbolic systems with weakly linearly degenerate characteristic fields (WLD), the classical solution exists globally in time provided that the initial data are suitably small and decay at the infinity [4]. This tells us that the formation of singularity depends strongly on the character of characteristics of the system. For the initial-boundary value problem, the situation is quite different. Even for quasilinear hyperbolic systems of diagonal form with linearly degenerate characteristic fields (a special case of WLD), the solution itself may blow up in a finite time (see [5]). Then it is natural to ask whether

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the classical C^1 solution exists globally in time when the solution itself can be controlled. The answer of this question is positive for homogeneous reducible hyperbolic systems and, more generally, for homogeneous rich hyperbolic systems with linearly degenerate characteristic fields, for which Lax transformation can be used to simplify the equations [5, 6]. In this paper, we want to extend this result to general homogeneous hyperbolic systems of diagonal form with linearly degenerate characteristic fields.

We consider then the following strictly hyperbolic system of diagonal form :

$$\frac{\partial u_i}{\partial t} + \lambda_i(u)\frac{\partial u_i}{\partial x} = 0 \quad (i = 1, ..., n) \tag{1.1}$$

for $t \geq 0$ and $x \in [0, 1]$, where $u = (u_1, ..., u_n)$, the eigenvalues $\lambda_i(u)$ $(i = 1, ..., n)$ are supposed to be smooth and satisfy

$$\lambda_1(u) < ... < \lambda_m(u) < 0 < \lambda_{m+1}(u) < ... < \lambda_n(u) \tag{1.2}$$

for any given u on the domain under consideration. Moreover, suppose $\lambda_i(u)$ $(i = 1, ..., n)$ are all linearly degenerate in the sense of P.D.Lax, i.e.,

$$\frac{\partial \lambda_i(u)}{\partial u_i} \equiv 0 \quad (i = 1, ..., n). \tag{1.3}$$

The system (1.1) is supplemented by the initial conditions

$$t = 0 : \quad u_i = \varphi_i(x) \quad (i = 1, ..., n) \tag{1.4}$$

for $x \in [0, 1]$ and the boundary conditions

$$x = 0 : \quad u_k = g_k(t, u_1, ..., u_m) \quad (k = m + 1, ..., n), \tag{1.5}$$

$$x = 1 : \quad u_j = g_j(t, u_{m+1}, ..., u_n) \quad (j = 1, ..., m) \tag{1.6}$$

for $t \geq 0$, where φ_i and g_i $(i = 1, ..., n)$ are all C^1 functions with respect to their arguments, and the conditions of C^1 compatibility are supposed to be satisfied at points $(t, x) = (0, 0)$ and $(0, 1)$ respectively.

When λ_i $(i = 1, ..., n)$ are constants, it has been proved (see [5]) that the initial-boundary value problem (1.1)-(1.6) always admits a unique global C^1 solution. This shows that without the nonlinearity of system, the nonlinear boundary conditions can not lead to the formation of singularity; otherwise, the C^1 solution may blow up in a finite time. The goal of this paper is to prove that if the C^1 solution to the initial-boundary value problem (1.1)-(1.6) blows up in a finite time, then the solution itself must tend to the infinity at the starting point of singularity. This kind of blow up phenomenon is similar to the breakdown of C^1 solution to the Cauchy problem for inhomogeneous reducible quasilinear hyperbolic systems of diagonal form with linearly degenerate characteristic fields (see [7] or Chapter 2 in [3]).