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TRAVELING WAVE FRONTS OF A DEGENERATE PARABOLIC EQUATION WITH NON-DIVERGENCE FORM

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Dedicated to the 80th birthday of Professor Zhou Yulin

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Abstract We study the traveling wave solutions of a nonlinear degenerate parabolic equation with non-divergence form. Under some conditions on the source, we establish the existence, and then discuss the regularity of such solutions.

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1. Introduction

This paper is concerned with the traveling wave fronts of the following nonlinear degenerate equation with non-divergence form

$$\frac{\partial u}{\partial t} = u^m \triangle u + u^n f(u), \qquad x \in \mathbb{R}^N, t \in \mathbb{R}^+, \tag{1.1}$$

where $m \geq 1$, $n > 0$ and f is continuously differentiable. Such an equation is quite different from the well-known porous medium equation with an absorption

$$\frac{\partial u}{\partial t} = \Delta u^p + u^q f(u), \qquad (p > 1, q > 0) \tag{1.2}$$

although it can be transformed into an equation like (1.1), with the exponent $m = \dfrac{p-1}{p}$ which falls into the interval $(0,1)$. During the past decades, the equations whose principal parts are in divergence form, like (1.2), have been deeply investigated. However, as far as we know, there are only a few works devoted to the equations whose principal parts are not in divergence form like (1.1). Among the earliest works in this respect, it is worthy to mention the work [1] by Allen, who did discuss such kind of equation with $m = 1$ in one dimensional case, modeling the diffusive process for biological species. It was Friedman and McLeod [2] who studied the blow-up properties of solutions for the equation with $m = 2$, $n = 3$ in multi-dimensional case. We may also mention the work

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[3] by Passo, where the basic existence, uniqueness and the properties of solutions are investigated in detail for the case $m = 1$. Recently, Wang, Wang and Xie [4] studied the equation for any $m > 1$ with $n = m + 1$, and discussed the global existence and blow-up properties of solutions. Furthermore, we point out that Bertsch has obtained several important results on the similar equations like (1.1) or (1.2), see [5–7].

In this paper, we are much interested in the discussion of the traveling wave solutions of the equation (1.1) with $m \geq 1$ and $n > 0$. For the same question about the degenerate or non-degenerate diffusion equations whose principal parts are in divergence form, we refer to [8–13]. First, we introduce the following

Definition *A function $u(z) \in C(\mathbb{R})$ with $z = \gamma \cdot x + t$ for some $0 \neq \gamma \in \mathbb{R}^N$ is called a traveling wave front of the equation (1.1) if there exist $-\infty \leq z_l < z_r \leq +\infty$ such that*

(i) *$u(z) \in C^2(z_l, z_r)$ and satisfies*

$$u' = |\gamma|^2 u^m u'' + u^n f(u), \qquad \forall z \in (z_l, z_r);$$

(ii) *$u(z_l) = \theta_l$, $u(z_r) = \theta_r$, where θ_l and θ_r are zero or the zero points of $f(u)$;*

(iii) *$u(z)$ is strictly monotone in the interval (z_l, z_r), $u(z) = \theta_l$ for $z \in (-\infty, z_l)$ and $u(z) = \theta_r$ for $z \in (z_r, +\infty)$;*

(iv) *If $u(z_l) < u(z_r)$, then $u'(z_r) = 0$, while if $u(z_l) > u(z_r)$, then $u'(z_l) = 0$.*

Furthermore, if $u'_+(z_l) = u'_-(z_r) = 0$, we call $u(z)$ a smooth traveling wave front, where u'_+ and u'_- denote the right and the left derivative of u.

To discuss the traveling wave fronts, let us first change the form of the equation. Let $p = u'$ and $c = \dfrac{1}{|\gamma|^2}$, the wave speed. Then for $z \in \{z \in (z_l, z_r) : u(z) > 0\}$, we get that

$$\begin{cases} u' = p, \\ p' = cu^{-m}p - cu^{n-m}f(u). \end{cases} \tag{1.3}$$

As we did for the equation whose principal part is in divergence form, we consider the following two typical cases

$$f(1) = 0, f'(1) < 0, \text{ and } f(s) > 0 \text{ for } s \in [0, 1), \tag{H1}$$

and

$$f(0) < 0, f(1) = 0, f'(1) < 0, f(u) < 0 \text{ for } s \in (0, a) \text{ and } f(s) > 0 \text{ for } s \in (a, 1), \tag{H2}$$

where a is a given number in $(0, 1)$. First, in Section 2 we discuss the case for f satisfying (H1). Different from the equation (1.2), see [14], there is no minimal wave speed for the solutions of the equation (1.1). In other words, for any c, there always exists a traveling wave front with the wave speed c for equation (1.1). Then in Section 3, we study the case with f changing sign, namely, the case for f satisfying (H2). As it was shown in [15], there exists one and only one wave speed c^* such that the equation