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NONLINEAR SCHRÖDINGER EQUATIONS WITH VARIABLE COEFFICIENTS–BLOWUP*

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1. Introduction

This paper concerns the following Cauchy problem for the nonlinear Schröinger equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = i\left\{\operatorname{div}\{f(x)\nabla u\} + k(x)|u|^2 u\right\}, \quad x \in \mathbb{R}^2, \quad t \geq 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where f and k are appropriately smooth real-valued functions on $\mathbb{R}^2$ and $u \in \mathbb{C}^n$.

The authors review their recent results ([1]) on the blow-up properties of the solutions. The reader is referred to our papers [1] for further references.

Throughout this paper, $Q_{L,K}$ is the unique radially symmetric (ground state) solution of

$$L\Delta Q + K|Q|^2 Q = Q, \quad \text{in } \mathbb{R}^2. \tag{2}$$

(See [2] for existence and [3] for uniqueness). δ_x denotes the Dirac measure at $x \in \mathbb{R}^2$.

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2. Blow-up properties

According to the results in [4], a natural question arises: How small is the L^2-norm of the initial data for global existence? The answer can be found in Theorem 1. In the following, we suppose the existence and uniqueness of the solution and only focus on the behavior of blow-up properties. In particular, we will be referring to the following assumptions on the functions $f(x)$ and $k(x)$.

(H1) $0 < L \equiv \inf_{x\in\mathbb{R}^2} f(x) \le f(x) \le \sup_{x\in\mathbb{R}^2} f(x) < +\infty,\ \forall x \in \mathbb{R}^2$;

(H2) $|x\cdot\nabla f(x)| + |\nabla f(x)| \le C,\ \forall x \in \mathbb{R}^2$ for some $C > 0$;

(H3) there is x_0 such that $f(x_0) = L$.

(H1)′ $0 < \inf_{x\in\mathbb{R}^2} k(x) \le k(x) \le \sup_{x\in\mathbb{R}^2} k(x) \equiv K < +\infty,\ \forall x \in \mathbb{R}^2$;

(H2)′ $|x\cdot\nabla k(x)| + |\nabla k(x)| \le C,\ \forall x \in \mathbb{R}^2$ for some $C > 0$;

(H3)′ there is x_0 satisfying (H3) such that $k(x_0) = K$.

Our main results are as follows.

Theorem 1 (L^2-concentration phenomenon) *Assume that $f(x)$ and $k(x)$ satisfy* (H1)-(H2) *and* (H1)′-(H2)′ *respectively. Let $u(t)$ be a blow-up solution of the Cauchy problem* (1) *and T its blow-up time. Then*

(i) *there is $x(t) \in \mathbb{R}^2$ such that $\forall R > 0$*

$$\liminf_{t\uparrow T} \int_{|x-x(t)|<R} |u(t,x)|^2 dx \ge \|Q_{L,K}\|_{L^2}^2; \tag{3}$$

(ii) *there is no sequence $\{t_n\}$ such that $t_n \uparrow T$ and $u(t_n)$ converges in $L^2(\mathbb{R}^2)$ as $n \to +\infty$.*

Theorem 2 (L^2-concentration (radial case)) *Let $f(x)$ and $k(x)$ be radial with respect to x_0 i.e., $f(x) = f(|x-x_0|)$ and $k(x) = k(|x-x_0|)$, satisfying* (H1)-H(2) *and* (H1)′-(H2)′ *respectively, $u(t)$ the blow-up solution with radial (w.r.t. x_0) initial data u_0, T its blow-up time. Assume in addition that there is $\rho_0 > 0$ such that for $|x-x_0| < \rho_0$,*

$$(x-x_0)\cdot\nabla k(x) \le 0 \le (x-x_0)\cdot\nabla f(x). \tag{4}$$

Then the following two properties (A) and (B) *are equivalent:*

(A) $|u(t,x)|^2 \to \|u_0\|_{L^2}^2 \delta_{x_0}$ *in the distribution sense as $t \uparrow T$;*

(B) $|x-x_0|u_0 \in L^2(\mathbb{R}^2)$ *and* $\lim_{t\uparrow T} \||x-x_0|u(t)\|_{L^2} = 0$.

Theorem 3 (Existence of blow-up solutions) *Suppose $f(x)$ and $k(x)$ satisfy* (H1)-H(3) *and* (H1)′-(H3)′ *respectively. Assume in addition that*

$$\mathrm{curl}\left(\frac{x-x_0}{f(x)}\right) = 0 \quad (\textit{integrability condition}), \tag{5}$$

$$(x-x_0)\cdot\nabla f(x) \ge 0 \quad \text{for all } x \in \mathbb{R}^2 \text{ or} \tag{6}$$

$$\text{there is } \rho_0 > 0 \text{ such that } (x-x_0)\cdot\nabla f(x) > 0 \quad \text{for } 0 < |x-x_0| < \rho_0, \tag{7}$$