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THE SELF-SIMILAR SOLUTION FOR GINZBURG-LANDAU EQUATION AND ITS LIMIT BEHAVIOR IN BESOV SPACES

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Abstract In this paper, we study the limit behavior of self-similar solutions for the Complex Ginzburg-Landau (CGL) equation in the nonstandard function space $E_{s,p}$. We prove the uniform existence of the solutions for the CGL equation and its limit equation in $E_{s,p}$. Moreover we show that the self-similar solutions of CGL equation converge, globally in time, to those of its limit equation as the parameters tend to zero.

Key Words Ginzburg-Landau equation; Schrödinger equation; self-similar solution; limit behavior.

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1. Introduction

In this paper, we consider the following Cauchy problem for the complex Ginzburg-Landau equation

$$\begin{aligned} &u_t - \varepsilon\Delta u - i\Delta u + (a+ib)|u|^\alpha u = 0, \\ &u(0,x) = u_0(x), \end{aligned} \tag{1}$$

where $\varepsilon > 0$, $a \in \mathbf{R}$, $b \in \mathbf{R}$, $u(x,t)$ is a complex-valued function on $\mathbf{R}^n \times \mathbf{R}^+$. If we set $\varepsilon = 0$ or $\varepsilon = 0$, $a = 0$, the equation (1) formally becomes

$$v_t - i\Delta v + (a+ib)|v|^\alpha v = 0, \qquad v(0,x) = v_0(x), \tag{2}$$

or

$$v_t - i\Delta v + ib|v|^\alpha v = 0, \qquad v(0,x) = v_0(x). \tag{3}$$

An essential problem among (1), (2) and (3) is that: whether the solutions of (1) converge to those of (2), (3) as the parameter $\varepsilon \to 0^+$ or $\varepsilon \to 0^+$, $a \to 0$. Recently in [1], B.Wang gave a positive answer when the initial data in the energy spaces L^2 or H^1. He pointed out that for any fixed $T > 0$, the solutions of (1) converge in $C(0,T;H^s)$,

$s=0,1$. In this paper, we consider the case of self-similar solutions for (1)-(3). First of all, we observe that if $u(x,t)$ solves (1)-(3), then

$$D_\lambda u(x,t)=\lambda^{\frac{2}{\alpha}}u\left(\lambda x,\lambda^2 t\right), \tag{4}$$

is also a solution of (1)-(3) with initial data

$$u_{0\lambda}(x)=\lambda^{\frac{2}{\alpha}}u_0(\lambda x). \tag{5}$$

One recalls the solution u is self-similar if it satisfies

$$u(x,t)=D_\lambda u(x,t) \tag{6}$$

for any $(x,t)\in\mathbf{R^n}\times\mathbf{R}^+$ and $\lambda>0$, it's straightforward to verify $u(x,t)$ is a self-similar solution if and only if

$$u(x,t)=t^{-\frac{1}{\alpha}}u\left(\frac{x}{\sqrt{t}},1\right)=t^{-\frac{1}{\alpha}}W\left(\frac{x}{\sqrt{t}}\right), \tag{7}$$

for some $W:\mathbf{R^n}\to\mathbf{C^n}$ called the profile of the self-similar solution. Therefore, the equations (1)-(3) can be studied through a nonlinear elliptic equation on W. But these nonlinear elliptic equations are always complicated and are difficult to solve. On the other hand, one sees from (7) that the initial data of the self-similar solution have to verify

$$u_0(x)=\lambda^{\frac{2}{\alpha}}u_0(\lambda x), \qquad \forall\lambda>0. \tag{8}$$

For example, $u_0(x)=\frac{\Omega(x')}{|x|^{\frac{2}{\alpha}}}$, $x'=\dfrac{x}{|x|}$, Ω is a function on unit sphere. This leads to another method to treat self-similar solutions of CGL or NLS. Indeed, one chooses a suitable Banach space B as the work space, the well-posedness in B means the initial data in (8) develop into self-similar solutions. However, since such data never belong to any homogeneous Sobolev space, it is not easy to obtain the existence of self-similar solution. Recently, many authors have interests in this area and have done some works as well. For instance, by introducing the nonstandard function space, the existence of the self-similar solution for a class of data in (8) was established. Moreover the self-similar solutions were also used as describing the long-time behavior of the other solutions in a better way. For the details, one refers to [2-5] for nonlinear Schrödinger equation, refers [6-8] for nonlinear wave equation and refers [9], [2] for nonlinear heat equation and NS equation.

In our previous work[10], we dealt with the self-similar solution of CGL equation in the space of the type

$$X_{s,p}=\left\{u;\sup_{t>0}t^{\beta(s,p)}\|u(t)\|_{\dot{H}^{s,p}}<\infty\right\}, \tag{9}$$

and proved the self-similar solutions of CGL equation converge to the corresponding limit Schrödinger equation as the parameters tend to 0 provided that the dimension $1\le n\le 5$. In (9), $\beta(s,p)$ is chosen to preserve the scaling (4) invariance.