J. Partial Diff. Eqs. **15**(2002), 28–34
©International Academic Publishers Vol.15, No.1

SEMILINEAR ELLIPTIC EQUATIONS WITH SINGULARITY ON THE BOUNDARY*

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(Received Jun. 10, 2001; revised Dec. 14, 2001)

Abstract In this paper, we consider the existence and nonexistence of positive solutions to semilinear elliptic equation $-\Delta u = K(x)(1-|x|)^{-\lambda}u^q$ in the unit ball B with 0-Dirichlet boundary condition. Our main tools are based on the interior estimates of the Schauder type, the Schauder fixed point theorem and the pointwise estimates for Green functions.
Key Words Singularity; semilinear equation; positive solution.
2000 MR Subject Classification 35J65
Chinese Library Classification O175.6

1. Introduction

Let B be the unit ball $\{x \in R^n : |x| < 1\}$ in R^n and consider the semilinear elliptic problem

$$\begin{cases} -\Delta u = K(x)(1-|x|)^{-\lambda}u^q, & x \in B, \\ u(x) > 0, & x \in B, \\ u(x) = 0, & x \in \partial B, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $q > 1$ and $K(x)$ is a given nonnegative α-Hölder continuous function on $\overline{B}$. As a matter,of course, this kind of problems which allow $\lambda \le 0$ has been investigated extensively.

When $K(\cdot)$ is a given nonnegative continuous radial function on $\overline{B}$, this problem was already studied by Seuba-Ebihare-Furusho [1] within the framework of the theory of ODE. They obtained the existence of positive radial solutions in $C^2(B) \cap C^1(\overline{B})$ for the case $0 < \lambda < 2$ and $1 < q < (n+2)/(n-2)$. Hayashida-Nakatani [2] also studied some similar problems and discussed some mathematical backgrounds for (1.1). In a recent paper [3], Hashimoto-$\hat{O}$tani studied this problem by the variational method. They showed the existence of positive radial solutions in $C^2(B) \cap C^1(\overline{B})$ for the case

*Project supported by the NNSF of China (No. 10071080) and NNSFC for Young scholars (No. 10101024)

$0 < \lambda < 1 + (q+1)/2$ and $1 < q < (n+2)/(n-2)$. Meanwhile they obtained the nonexistence theorem of positive solutions in $C^2(B) \cap C^1(\overline{B})$ for the case $\lambda \geq 1+q$.

As was pointed in [3], it would be interesting to investigate the existence of (not necessarily classical) positive solutions of (1.1) for the case $1 + (1+q)/2 \leq \lambda < 1+q$.

The main purpose of this paper is devoted to the existence and nonexistence of positive solutions in $C^0(\overline{B}) \cap C^{2,\alpha}(B)$ of (1.1) from the viewpoint of the theory of nonlinear PDE. However, our method can deal with the sharper singular case (i.e., $0 < \lambda < 1 + \alpha + (q-1)\beta$, for some $0 < \alpha \leq \beta,\ 0 < \beta < 1$) than those in Hashimoto-Ôtanni [3]. Our argument is based on the interior estimates of the Schauder type, the Schauder fixed point theorem and the pointwise estimates for Green function. However, our argument does not require the symmetry of both $K(\cdot)$ and the solution. Moreover, we have droped out the subcritical condition $q < (n+2)/(n-2)$.

The main results are stated in the next section, and their proofs will be given in Section 3 and Section 4.

2. Main Result

Throughout this paper, the following condition will be imposed on $K(\cdot)$

$$(K_\alpha)\begin{cases} K(x) \in C^\alpha(\overline{B}), \\ K(x) \geq 0,\ x \in B, \\ K(x) > 0,\ x \in \partial B, \end{cases} \tag{2.1}$$

where $0 < \alpha \leq \beta$ for some $0 < \beta < 1$.

The main results of this paper are the following two theorems.

Theorem 2.1 (Existence theorem)*Let $K(\cdot)$satisfy condition (K_α) and $q > 1, 0 < \lambda < 1 + \alpha + (q-1)\beta$, then (1.1) has at least one positive solution $u(x)$ belonging to $C^0(\overline{B}) \cap C^{1,\alpha}(B)$. Furthermore, there exist positive constants c_1, c_2, c_3 and $\epsilon (0 < \epsilon < 1)$ such that*

$$|u|^{(-\beta)}_{0,\alpha;B} \leq c_1,\ |u|^{(-\beta)}_{2,\alpha;B} \leq c_2 c_1^q |K|_{0,\alpha;B},$$

$$u(x) \geq c_3(1 - |x|),\ \forall\ x \in B_{1-\epsilon},$$

where B_r is an open ball with radius r centered at origin.

Remark 2.1 As pointed out in the introduction, we do not need the subcritical condition $q < (n+2)/(n-2)$.

Remark 2.2 If we have $\alpha = \beta \to 1$, then $\overline{q} := 1+\alpha+(q-1)\beta \to 1+q$. Combining this with a nonexistence result of Hayashimoto-Ôtani [3], we know that our result, in some sense, is essentially optimal.

Remark 2.3 For the case of $\alpha = 0$, i.e., $K(\cdot)$ only continuous, the existence of (not necessary classical) positive solutions of Dirichlet problem (1.1) is still open.