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BERNSTEIN-JÖRGENS THEOREM FOR A FOURTH ORDER PARTIAL DIFFERENTIAL EQUATION*

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Abstract We introduce a metric, conformal to the affine metric, on a convex graph, and consider the Euler equation of the volume functional. We establish a priori estimates for solutions and prove a Bernstein-Jörgens type result in the two dimensional case.
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1. Introduction

In this paper we study locally uniformly convex solutions of fourth order elliptic equations of the form

$$L[u] = U^{ij} w_{ij} = f, \tag{1.1}$$

in the n-dimensional Euclidean space, $\mathbf{R}^n$, where (U^{ij}) is the cofactor matrix of the Hessian matrix $(u_{ij}) = D^2u \geq 0$, $w = [\det D^2 u]^\alpha$, $\alpha \neq 0$ is a constant, and f is a given function in $\mathbf{R}^n$. The operator L is the Euler operator (up to a constant) of the functional

$$J(u) = \int [\det D^2 u]^{1+\alpha}. \tag{1.2}$$

Let $\mathcal{M} = \{(x, u(x)) \mid \ x \in \mathbf{R}^n\}$ be a locally uniformly convex hypersurface, given by the graph of u. We introduce a metric g on M, defined by

$$g_{ij} = \rho u_{ij}, \tag{1.3}$$

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where $\rho = [\det D^2 u]^{(1+2\alpha)/n} > 0$. Then (1.2) is the volume functional of the metric g. Note that (1.1) can also be written in the form (suppose $f = 0$)

$$\Delta_g \rho = 0, \tag{1.4}$$

where Δ_g is the Laplace-Beltrami operator with respect to the metric g.

There has been a growing interest in recent years in functionals involving curvatures of a hypersurface (or manifold). Well-known examples are the Willmore functional [1,2]

$$\int_{\mathcal{M}} H^2 d\sigma, \tag{1.5}$$

the functional proposed by Calabi [3-5]

$$\int_{\mathcal{M}} S^2 d\sigma \tag{1.6}$$

and the affine surface area functional [6,7]

$$\int_{\mathcal{M}} K^{1/(n+2)} d\sigma, \tag{1.7}$$

where H, S, K are respectively the mean curvature, the scalar curvature, and the Gauss curvature, and $d\sigma$ is the volume element on $\mathcal{M}$. The Euler equations of these functionals are strongly nonlinear fourth order partial differential equations.

Our knowledge on higher order nonlinear partial differential equations is limited up to date, although there are some isolated results. The study of the functional (1.2) may help to understand other functionals such as (1.5)-(1.7). Note that the metrics g in (1.3) are conformal to each other for different α. If $\alpha = -\frac{1}{2}$, the metric g in (1.3) is called the Schwarz-Pick metric [8]. When $\alpha = -\frac{n+1}{n+2}$, the metric

$$g_{ij} = g_{ij}^a = [\det D^2 u]^{-1/(n+2)} u_{ij} \tag{1.8}$$

is the affine metric (Berwald-Blaschke metric). In this case the equation (1.1) is the affine maximal surface equation for $f = 0$, and the affine mean curvature equation for general f. In [7] we proved interior estimates and solved the Bernstein problem in dimension two for the affine maximal surface equation.

In this paper we study the equation (1.1) with positive exponent $\alpha > 0$. We will first derive a priori estimates (Section 2) and then prove the Bernstein-Jörgens theorem for the equation (1.1), with $f \equiv 0$, in two dimensions (Section 3). In [9] Jörgens proved that an entire convex solution to the Monge-Ampère equation

$$\det D^2 u = 1 \tag{1.9}$$

must be a quadratic function if $n = 2$. Jörgens' result was extended to high dimensions by Calabi for $3 \le n \le 5$ and Pogorelov for all $n \ge 2$, see [10]. Jörgens' result can also