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REGULARITY OF THE SOLUTION TO THE DIRICHLET PROBLEM IN MORREY SPACES

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Abstract The aim of this paper is to study Morrey regularity of the solution to the Dirichlet problem for second order elliptic equation of the form

$$Lu = -(a_{ij}u_{x_i} + b_j u)_{x_j} + du = e + (f_j)_{x_j}$$

in a bounded open subset of $R^n (n \geq 3)$, where b_j, d, e, f_j are assumed to be in some Morrey spaces.
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1. Introduction

In this paper, we study the following second order elliptic equation in divergence form

$$-(a_{ij}u_{x_i} + b_j u)_{x_j} + c_j u_{x_j} + du = e + (f_j)_{x_j} \tag{1}$$

in Ω, a bounded open subset of R^n $(n \geq 3)$, where $a_{ij}(x)$ are bounded, symmetric, measurable and satisfy uniformly elliptic condition.

Sobolev spaces introduced by S. L. Sobolev in the middle 1930's provide an effective way to study the solvability and regularity of elliptic equations. By using Sobolev spaces, we can look for solutions in function classes with wider area and study their regularity. Such solution is often called "weak solution". When the weak solution has good regularity, it is actually a classical one. Up to 1950-60's, a lot of outstanding results about the equation (1) had been gotten after many people's systematic researches (see e.g. [1-4]). Among them, W. Littman, G. Stampaachia and H. F. Weinberger introduced Green's function in [4], and provided a very important method for studying the regularity of the equation (1).

When studying (1), if the lower order and nonhomogeneous terms are considered, they must be demanded to have high integrability by classical results (see e.g. [1,2]).

In order to study the local regularity of the solutions of second order elliptic equations, C. B. Morrey introduced Morrey spaces $L^{p,\lambda}$ in [5]. The integrability of $L^{p,\lambda}$ is not greater than p. After that, some people have been studying the regularity of solutions of the equation (1) and their results can be seen in [6-12], etc.

Now we give the main result in this paper.

We consider the following Dirichlet problem

$$\begin{cases} Lu = -(a_{ij}u_{x_i} + b_j u)_{x_j} + du = e + (f_j)_{x_j}, & in\,\Omega, \\ u = 0, & on\,\partial\Omega \end{cases} \tag{2}$$

in Ω, a bounded open subset of $\mathrm{R^n}$ ($\mathrm{n} \geq 3$), where

$$\begin{cases} a_{ij} \in L^\infty(\Omega),\, a_{ij} = a_{ji}, \quad i,j = 1,\cdots,n, \\ \exists \nu > 0 :\ a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2\, a.e.\, in\,\Omega,\ \forall \xi \in \mathrm{R^n}, \\ b_j \in L^{2,\mu}(\Omega), \quad n-2 < \mu < n,\ j = 1,\cdots,n, \\ d \in L^{2_*,\mu}(\Omega), \quad n-2 < \mu < n,\ 2_* = \frac{2n}{n+2}, \\ e \in L^{2_*,\tau}(\Omega), \quad \frac{n(n-2)}{n+2} < \tau < n, \\ f_j \in L^{2,\lambda}(\Omega), \quad 0 < \lambda < n-2,\ j = 1,\cdots,n. \end{cases} \tag{3}$$

Throughout this paper, we shall assume that there exists a positive constant A such that $|\Omega \cap B_r(x)| \geq Ar^n$ for all $x \in \Omega$ and for all $r \leq \operatorname{diam}\Omega$.

We get our result which is contained in

Theorem 1.1 *If u is a weak solution of Dirichlet problem (2), then for any* $0 < \lambda < \dfrac{n(n-2)}{n+2}$, *there exists a constant $C = C(n,\nu,\lambda,\mu,\tau,\Omega)$ such that*

$$\|u\|_{L_w^{p_\lambda,\lambda}(\Omega)} \leq C(\|bu\|_{2,\lambda} + \|du\|_{2_*,\lambda} + \|e\|_{2_*,\tau} + \|f\|_{2,\lambda}), \tag{4}$$

where $\dfrac{1}{p_\lambda} = \dfrac{1}{2} - \dfrac{1}{n-\lambda}$.

We wish to point out that our result improves that in [10] and [12]. The definitions of the above spaces can be seen in Section 2.

2. Definitions and Known Results

In this section, we give some definitions and known results.

Definition 2.1 *Let* $1 \leq p < +\infty$, $0 \leq \lambda \leq n$, $\|f\|_{p,\lambda}^p = \sup\limits_{\substack{r>0\\x\in\Omega}} r^{-\lambda}\displaystyle\int_{\Omega_r(x)} |f(x)|^p dx$, *where $\Omega_r(x) = \Omega \cap B_r(x)$. The subset of those functions of $L^p(\Omega)$ for which $\|f\|_{p,\lambda} < +\infty$ will be called the Morrey space $L^{p,\lambda}(\Omega)$.*

Definition 2.2 *Let $1 \leq p < +\infty$, $0 \leq \lambda \leq n$. The set of those measurable f such that $\sup\limits_{t>0} t^p|\{x \in \Omega : |f(x)| > t\} \cap B_r(x)| \leq Cr^\lambda$ for some $C > 0$ independent of $r > 0$ and $x \in \Omega$ will be called the weak Morrey space $L_w^{p,\lambda}(\Omega)$.*