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SEMICLASSICAL LIMIT OF NONLINEAR SCHRÖDINGER EQUATION (II)

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Abstract In this paper, we use the Wigner measure approach to study the semiclassical limit of nonlinear Schrödinger equation in small time. We prove that: the limits of the quantum density: $\rho^\epsilon =: |\psi^\epsilon|^2$, and the quantum momentum: $J^\epsilon =: \epsilon Im(\overline{\psi^\epsilon}\nabla\psi^\epsilon)$ satisfy the compressible Euler equations before the formation of singularities in the limit system.

Key Words Schrödinger, compressible Euler, Wigner transformation, Wigner Measure.

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1. Introduction

In this paper, we consider the local in time semiclassical limit of the following nonlinear Schrödinger equation in three space dimension:

$$\begin{cases} i\epsilon\partial_t\psi^\epsilon = -\frac{\epsilon^2}{2}\triangle\psi^\epsilon + V^\epsilon\psi^\epsilon, \quad V^\epsilon = g(|\psi^\epsilon|^2), & x\in\mathbb{R}^3, t\geq 0, \\ \psi^\epsilon(t=0,x) = \sqrt{\rho_0^\epsilon(x)}\exp(\frac{i}{\epsilon}S^\epsilon(x)), \end{cases} \tag{1.1}$$

where ψ^ϵ denotes the condensate wave function in the quantum mechanics, and ϵ is the normalized Planck constant.

Equations of type (1.1) have been proposed as multiparticle approximations in the mean-field theory of Quantm Mechanics, when one considers a large number of quantum particles acting in unison and takes into account only a finite number of particle-particle intereactions. It is a fundamental principle in quantum mechanics that: when the time and distance scales are large enough relative to the Planck's constant, the quantum density: $|\psi^\epsilon|^2$, and the quantum momentum: $\epsilon Im(\overline{\psi^\epsilon}\nabla\psi^\epsilon)$, will approximately obey the laws of classical, Newtonian mechanics. And the quantum-mechanical pressure disappears in the semiclassical limit, the isentropic compressible Euler equations are formally recovered from the nonlinear Schrödinger equation.

When $g'(\cdot) > 0$,, the phase function $S^\epsilon(x)$ is independent of ϵ, and the amplitude $\sqrt{\rho_0^\epsilon(x)}$ is given by the expansion: $\sum_{i=1}^N a_j(x)\epsilon^j + \epsilon^N r_N(x,\epsilon)$ with $\lim_{\epsilon\to 0}\|r_N(\cdot,\epsilon)\|_{H^s} =$

0 for s large enough, Grenier ([1]) obtained a similar expansion for the solution of (1.1) in small time. His main idea is that: instead of looking as usual at solution ψ^ϵ of the form:

$$\psi^\epsilon(t,x) = a^\epsilon(t,x)e^{\frac{iS(t,x)}{\epsilon}},$$

with S independent of ϵ, he looks for solution ψ^ϵ of the form:

$$\psi^\epsilon(t,x) = a^\epsilon(t,x)e^{\frac{iS^\epsilon(t,x)}{\epsilon}}, \tag{1.2}$$

where a^ϵ is again a complex-valued function. By plugging (1.2) to (1.1), separating the real and imaginary part, one can get the governing equations for a^ϵ and S^ϵ. Then the standard energy estimate for symmetric hyperbolic equations can be used to solve this problem. And in one space dimension with $V^\epsilon = (|\psi|^2 - 1)$, Jin, Levermore and Mclaughlin globally ([2]) solved the limit by the inverse scattering method.

This paper is a following one of [3]. As in [3], we consider the oscillatory initial data for (1.1). Here $S^\epsilon(x)$ depends on ϵ, and $\sqrt{\rho_0^\epsilon(x)}$ does not have the explicit expansion any more. Instead, we will assume some limits for ρ_0^ϵ and ∇S^ϵ, then study what kind of equations will be satisfied by the weak limits of $|\psi^\epsilon|^2$ and $\epsilon Im(\overline{\psi^\epsilon}\nabla\psi^\epsilon)$ in small time. The main idea of the proof is from [3], which is motivated by [4] and [5], also this idea is used by Marjolaine in her thesis on the convergence of scaled Schrödinger-Poisson equation to the incompressible Euler equation. Namely, we are going to study the Wigner transformation $f^\epsilon(t,x,\xi)$ to the solutions of (1.1):

$$f^\epsilon(t,x,\xi) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-i\xi y}\psi^\epsilon(t,x+\frac{\epsilon y}{2})\overline{\psi^\epsilon(t,x-\frac{\epsilon y}{2})}\,dy, \tag{1.3}$$

which was introduced by Wigner in 1932 in quantum mechanics.

Then trivial calculation shows that $f^\epsilon(t,x,\xi)$ satisfies the following equation:

$$\begin{cases} \partial_t f^\epsilon + \xi\cdot\nabla f^\epsilon + \theta[V^\epsilon]f^\epsilon = 0, \\ f^\epsilon(t=0,x,\xi) = f_I^\epsilon(x,\xi), \end{cases} \tag{1.4}$$

where $\theta[V^\epsilon]f^\epsilon(t,x,\xi)$ is the pseudo-differential operator:

$$\begin{aligned}&\theta[V^\epsilon]f^\epsilon(t,x,\xi)\\ &= \frac{i}{(2\pi)^d}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{V^\epsilon(t,x+\frac{\epsilon y}{2}) - V^\epsilon(t,x-\frac{\epsilon y}{2})}{\epsilon} f^\epsilon(t,x,\eta)e^{-i(\xi-\eta)y}\,d\eta\,dy.\end{aligned} \tag{1.5}$$

Formally passing $\epsilon \to 0$ in (1.4), we get

$$\begin{cases} \partial_t f + \xi\cdot\nabla_x f - E\nabla_\xi f = 0, \\ E = \nabla g(\rho), \quad \rho = \int_{\mathbb{R}^3} f(t,x,d\xi), \\ f(t=0,x,\xi) = f_0(x,\xi), \end{cases} \tag{1.6}$$