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SPECTRAL ASYMPTOTIC BEHAVIOR FOR A CLASS OF SCHRÖDINGER OPERATORS ON 1-DIMENSIONAL FRACTAL DOMAINS*

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Abstract In this paper, we study the spectral asymptotic behavior for a class of Schrödinger operators on 1-dimensional fractal domains. We have obtained, if the potential function is locally constant, the exact second term of the spectral asymptotics. In general, we give a sharp estimate for the second term of the spectral asymptotics.

Key Words Counting function; Schrödinger operator; Minkowski dimension.
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1. Introduction

Let Ω be an open set in $R^n (n \geq 1)$, with boundary $\Gamma = \partial\Omega$. We assume that Ω is non-empty and of finite volume (n-dimensional Lebesgue measure). We consider the following eigenvalue problem of Schrödinger operator:

$$\begin{cases} -\Delta u + \lambda V u = \mu u & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases} \tag{P}$$

where $\lambda > 0, V(x) \in C(\bar{\Omega})$, and Δ denotes the Dirichlet Laplacian on Ω. In fact, the scalar μ is said to be an eigenvalue of (P) if there exists $u \neq 0$ in $H_0^1(\Omega)$ which satisfies (P) in the distributional sense. It is well known that, the problem (P) has discrete eigenvalues if λ is given, which can be written in increasing order according to their finite multiplicities:

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_r \leq \cdots < +\infty \quad \text{with } \mu_r \to \infty \text{ as } r \to \infty \tag{1.1}$$

Let $E > 0, N(E, \lambda)$ denote the "counting function" of (P) associated with E, i.e. $N(E, \lambda) = \#\{k \geq 1, \mu_k \leq \lambda E\}$ is the number of eigenvalues of (P) less than λE, which is counted according to the multiplicities.

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We have known that in one-dimensional case (see [1]),

$$N(E,\lambda)=\left[\frac{1}{\pi}\int_{\{x\in\Omega|V(x)<E\}}(E-V(x))^{1/2}dx+o(1)\right]\lambda^{1/2},\quad \Omega\subset R^1 \tag{1.2}$$

as $\lambda\to\infty$.

In this paper, we are interested in the sharper asymptotic form of $N(E,\lambda)$ for Ω with fractal boundary. First, let us recall some results on the Weyl conjecture and the Weyl-Berry conjecture, which will help us to understand the main result in this paper.

Consider

$$\begin{cases}-\Delta u=\lambda u & \text{in } \Omega\\ u=0 & \text{on } \Gamma\end{cases} \tag{Q}$$

where Δ denotes the Dirichlet Laplacian defined in Ω. Let $\mu>0, N_0(\mu)=\#\{k\geq 1; 0<\lambda_k\leq\mu\}$, where λ_r is Dirichlet eigenvalue of (Q).

In 1912, H.Weyl[2] proved that if Ω has sufficiently smooth boundary, then

$$N_0(\mu)=\varphi(\mu)(1+o(1)) \text{ as } \mu\to+\infty \tag{1.3}$$

where $\varphi(\mu)$ is called the "Weyl term" and is given by

$$\varphi(\mu)=(2\pi)^{-n}B_n|\Omega|_n\mu^{n/2} \tag{1.4}$$

Here, $|\Omega|_n$ denotes the n-dimensional Lebesgue measure of Ω and B_n is the volume of unit ball in R^n. Furthermore, he conjectured that, in the "smooth" case,

$$N_0(\mu)=\varphi(\mu)-C_n|\Gamma|_{n-1}\mu^{(n-1)/2}+o(\mu^{(n-1)/2}) \text{ as } \mu\to+\infty \tag{1.5}$$

where C_n is a positive constant depending only on n. (Here $|\Gamma|_{n-1}$ denotes the $(n-1)$-dimensional Lebesgue measure of Γ).

An important step on the way to the Weyl's conjecture was made by R.T.Seeley[3], and then by Pham The Lai [4]. They showed that for Γ is C^∞ smooth, then

$$N_0(\mu)=\varphi(\mu)+O(\mu^{(n-1)/2}) \text{ as } \mu\to+\infty \tag{1.6}$$

Further, V.Ja.Ivrii[5,6] and Melrose[7,8] have established (1.7) under some additional assumption, i.e. the Weyl's conjecture is true under some conditions.

How about the situation if the boundary Γ is non-smooth? The physicist Michael V.Berry[9] made the following conjecture: If the boundary $\Gamma=\partial\Omega$ is "fractal" with Hausdorff dimension $H\in(n-1,n)$ and H-dimensional Hausdorff measure $\mathcal{H}(H;\Gamma)$, then

$$N_0(\mu)=\varphi(\mu)-C_H\mathcal{H}(H;\Gamma)\mu^{H/2}+o(\mu^{H/2}) \text{ as } \mu\to+\infty \tag{1.7}$$

where C_H is a positive constant depending only on H. Berry even conjectured that $C_H=(4(4\pi)^{H/2}\Gamma(1+H/2))^{-1}$, where $\Gamma(s)$ denotes the classical gamma function.