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REGULARITY RESULTS FOR NONLINEAR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS UNDER WEAK ELLIPTICITY CONDITIONS*

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Abstract We prove $C^{1,\alpha}$ almost everywhere regularity for weak solutions in the space $W^{1,k}(\Omega, R^N)$ of the systems $-D_\alpha A^i_\alpha(x,u,Du) = B^i(x,u,Du)$ under the weak ellipticity condition $\int A(x_0,u,p+D\Phi)\cdot D\Phi dy \geq \lambda\int(|D\Phi|^2+|D\Phi|^k)dy$.
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1. Introduction

The partial regularity of weak solution $u : \Omega \to R^N$ in Sobolev space $W^{1,k}(\Omega, R^N)$, $(k \geq 2)$ of second order nonlinear systems in the form

$$-D_\alpha A^i_\alpha(x,u,Du) = B^i(x,u,Du) \tag{1.1}$$

has been extensively studied by many authors during last two decades. We refer the readers to Giaquinta's book [1] for a complete survey on the subject. In almost all case, the following strong ellipticity condition is needed:

$$\frac{\partial A^i_\alpha}{\partial p^j_\beta}(x,u,p)\xi^i_\alpha\xi^j_\beta \geq \lambda(|\xi|^2+|\xi|^k), \quad \forall\xi, \quad \lambda>0 \tag{1.2}$$

although in the recent paper [2], the condition (1.2) is replaced by the uniformly strictly monotone condition

$$(A^i_\alpha(x,u,p)-A^i_\alpha(x,u,\bar{p}))(p^i_\alpha-\bar{p}^j_\beta) \geq \lambda(|p-\bar{p}|^2+|p-\bar{p}|^k) \tag{1.3}$$

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However, in the case of $k = 2$, Fuchs [3] is replaced (1.2) by a natural ellipticity condition

$$\int A(p + D\Phi) \cdot D\Phi dy \geq \lambda \int |D\Phi|^2 dy \tag{1.4}$$

and proved regularity results for (1.1), which extended the idea of Evans [4], Fusco and Hutchinson [5].

The aim of the present paper is to generalize the results of [3], but our argument is different from Fuchs. Since Fuchs used a direct method which depends on the technique of reverse Hölder inequality (See [1]), we will use a blow-up and a compactness method which was developed in [6] for the calculus of variations and in [7] for elliptic systems. For this purpose, we replace (1.2) by the so-called quasi-convexity conditions, originated from variations integrals in [4–6], for the equation (1.1):

$$\int A(x_0, u, p + D\Phi) \cdot D\Phi dy \geq \lambda \int (|D\Phi|^2 + |D\Phi|^k) dy \tag{1.5}$$

and use some idea in [6]. Even if $k = 2$, our results (Theorems 2.1 and 2.2) include something new that cannot be obtained with the method in [3] (See Remark 2.3 below).

We would like to point out that Zhang [8] obtained the existence results for (1.1) just under (1.4) instead of (1.2), and listed many examples which satisfy (1.4) but not (1.2). By the way, in the paper [9], replacing (1.2) by asymptotic ellipticity, Jian obtained the Lipschitz continuity for weak solutions of (1.1).

2. Hypotheses and Statements of the Main Results

Suppose that $N \geq 1$, $n \geq 2$, are integers and Ω is a bounded open set in R^N. Denote $B(x, r) = \{y \in R^N : |y - x| < r\}$, $B = B(0, r)$ and $B = B_1$. Let $k \geq 2$. Consider the following second order nonlinear systems of PDE:

$$-D_\alpha A_\alpha^i(x, u, Du) = B^i(x, u, Du), \quad i = 1, 2, \cdots, N \tag{2.1}$$

where A_α^i, B^i ($\alpha = 1, 2, \cdots, n; i = 1, 2, \cdots, N$): $\Omega \times R^N \times R^{nN} \to R$ satisfy the following conditions: there exist a non-increase positive function $\lambda(\cdot)$ and non-decrease $\Lambda(\cdot)$ and $a(\cdot)$ and a positive function $\lambda(\cdot)$ such that for all $(x, u, p) \in \Omega \times R^N \times R^{nN}$

(F_1) $\int_B A_\alpha^i(x, u, p + \varphi(y)) D_\alpha \varphi^i(y) dy \geq \lambda(|u|) \int_B (|D\varphi|^2 + |D\varphi|^k) dy$, $\forall \varphi \in W_0^{1,k}(B, R^N) \cap L^\infty(B, R^N)$;

(F_2) $|A(x, u, p)| \leq \Lambda(|u|)(1 + |p|^{k-1})$, $A = (A_\alpha^i)$;

(F_3) $\dfrac{\partial A}{\partial p} \in C(\bar{\Omega} \times R^N \times R^{nN})$, $\left|\dfrac{\partial A}{\partial p}(x, u, p)\right| \leq \Lambda(|u|)(1 + |p|^{k-2})$;

(F_4) $|A(x, u, p) - A(y, v, p)| \leq \Lambda(|u| + |v|)(1 + |p|^{k-1})(|x - y| + |u - v|)^\sigma$ for some $0 < \sigma \leq 1$;