J. Partial Diff. Eqs. 13(2000), 235–242
©International Academic Publishers Vol.13, No.3

A NOTE ON THE P-LAPLACIAN EQUATION WITH SINGULAR COEFFICIENTS*

Zeng Youdong and Chen Zuchi
(Department of Mathematics, University of Science and Technology of China, Hefei 230026, Anhui, China)
(Received July 1, 1999; revised Dec. 23, 1999)

Abstract We consider the solvability of the Dirichlet problem in a unit n-ball for the p-Laplacian equation with singular coefficients which are singular not only as $|x|$ tends to 1 but also as $|x|$ tends to 0. The existence and regularity of positive radial solutions are proved under some conditions related to parameters p, τ, λ and q.
Key Words Positive radial solution; singular coefficient; p–Laplacian equation.
1991 MR Subject Classification 35J65,35B25.
Chinese Library Classification O175.25.

1. Introduction

Let B_1 be the unit ball centered at the origin in R^n. We consider the existence and regularity of positive radial solutions of the Dirichlet problem for p-Laplacian equation

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = a(|x|)|x|^{\tau}(1-|x|)^{-\lambda}|u|^{q-2}u, & \text{in } B_1\backslash\{0\} \\ u = 0, & \text{on } \partial B_1 \end{cases} \tag{1.1}$$

where $1 < p < n, \lambda \geq 0, \tau > -p, q > p, a : [0,1] \to [0,\infty)$ is continuous in [0,1], locally Hölder continuous in [0,1) and $a(1) \neq 0$.

The existence, nonexistence and multiple results for (1.1) with $\lambda = \tau = 0$ have been discussed by many mathematicians. Paper [1] treated (1.1) with $p = 2$ and the nonlinear term $g(x,u)$ in an arbitrary bounded domain with smooth boundary, in which the growth order of $g(x,u)$ in u is, roughly speaking, $|u|^{q-1}$ for $q \in (2, 2n/(n-2))$. Later, [2] proved, by the Pohozaev identity, that (1.1) has no nontrivial solution if $p = 2$ and $q \geq 2n/(n-2)$ provided $a(|x|)$ is decreasing. Among others [3] was concerned in the multiple solutions for (1.1).

Ni [4] concerned in (1.1) with $p = 2, \lambda = 0$ and $\tau > 0$ and extended q up to less than $2(n+\tau)/(n-2)$. Xu and Wu [5] extended the results of [4] to the case of $p > 1$. T.Seuba et al.[6] and Dalmasso [7] concerned in (1.1) for the case of $p = 2, \tau = 0$ and $\lambda > 0$. In this case, the coefficients of the right-hand side of (1.1) are singular as $|x| \to 1$. In

* The project supported by the Fund of Academia Sinica.

[6] and [7], q is also less than $2n/(n-2)$. Recently, Xuan and Chen [8] discussed (1.1) with $p>1, \tau \geq 0$ and $\lambda \geq 0$, and extended q up to less than $p(n+\tau)/(n-p)$.

In this note, we extend the results of [8] to the case of $p>1, \tau>-p$ and $\lambda \geq 0$. In this case, the coefficient of the right-hand side of (1.1) is singular not only as $|x| \to 1$ but also as $|x| \to 0$. All of the results for the case of $p>1$ in (1.1) may be regarded as a natural extension of the case $p=2$ in (1.1), i.e., the semilinear problem (see [9]).

Our proof for the existence of the weak solutions of (1.1) is based on the following Lemma and an extension of the embedding lemma in [8] (see Section 2).

Lemma 1.1 (Mountain pass lemma) (cf.[1]) *Let E be a real Banach space and $I \in C^1(E,R)$. Suppose that*

(I_1) I satisfies the Palais-Smale condition, i.e., any sequence $\{u_k\} \subset E$, for which $\{I(u_k)\}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$, possesses a convergent subsequence;

(I_2) $I(0)=0$ and there is a $u_0 \in E \backslash \{0\}$ such that $I(u_0) \leq 0$;

(I_3) there are constants $\rho \in (0, \|u_0\|)$ and $\alpha>0$ such that $I(u) \geq \alpha$ on $S_\rho = \{u \in E : \|u\| = \rho\}$.

Then I possesses a critical value $C \geq \alpha$. Moreover, C can be characterized as

$$C = \inf_{g \in \Gamma} \max_{u \in g[0,1]} I(u)$$

where

$$\Gamma = \{g \in C([0,1],E) : g(0)=0, g(1)=u_0\}$$

Using the Mountain pass lemma, we prove an existence theorem of positive radial weak solutions of the problem (1.1) under certain conditions on the parameters n,p,τ,λ, and q which will be given in Section 2 in detail. In Section 3, we show the $C^{1,\alpha}$-regularity of the obtained solution. In the sequel, C denotes a positive constant may varying line by line.

2. The Proof of Existence Theorem

Denote

$$E = \{u \in W_0^{1,p}(B_1) | u(x) \text{ is radially symmetric}\}$$

equipped with the norm

$$\|u\|_E = \left(\int_{B_1} |\nabla u|^p dx\right)^{1/p}$$

then E is a Banach space. Define f by $f(t)=t^{q-1}$ for $t>0, f(t)=0$ for $t \leq 0$ and $F(t) = \int_0^t f(s)ds$ for $t \in R$. Define the functional on E as

$$I(u) = \frac{1}{p}\|u\|_E^p - J(u), \quad u \in E \tag{2.1}$$

$$J(u) = \int_{B_1} \frac{a(|x|)|x|^\tau}{(1-|x|)^\lambda} F(u(x))dx, \quad u \in E \tag{2.2}$$