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GLOBAL $W^{2,p}$ $(2 \le p < \infty)$ SOLUTIONS OF GBBM EQUATIONS IN ARBITRARY DIMENSIONS

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Abstract This paper studies the initial-boundary value problem of GBBM equations

$$u_t - \Delta u_t = \operatorname{div} f(u) \tag{a}$$

$$u(x, 0) = u_0(x) \tag{b}$$

$$u|_{\partial\Omega} = 0 \tag{c}$$

in arbitrary dimensions, $\Omega \subset \mathbf{R}^n$. Suppose that $f(s) \in C^1$ and $|f'(s)| \le C(1+|s|^\gamma)$, $0 \le \gamma \le \dfrac{2}{n-2}$ if $n \ge 3$, $0 \le \gamma < \infty$ if $n = 2$, $u_0(x) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ $(2 \le p < \infty)$, then $\forall T > 0$ there exists a unique global $W^{2,p}$ solution $u \in W^{1,\infty}(0, T; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$, so the known results are generalized and improved essentially.

Key Words GBBM equation; initial-boundary value; global $W^{2,p}$ solution.

Classification 35Q.

1. Introduction

There are already many results $[1-7]$ on the existence and uniqueness of global solutions of the initial-boundary value problem for GBBM equations

$$u_t - \Delta u_t = \operatorname{div} f(u) \tag{1}$$

$$u(x, 0) = u_0(x) \tag{2}$$

$$u|_{\partial\Omega} = 0 \tag{3}$$

where $\Omega \subset \mathbf{R}^n$ is a smooth bounded domain. In [5-7] Chen Yunmei, Goldstein and Guo Boling et al. all studied global $W^{2,p}$ solutions of the problem (1)–(3) respectively, the results obtained by them are as follows: Assume that $\partial\Omega$ is sufficiently smooth, $f(s) \in C^2$, $f'(0) = 0$ and satisfies the hypothesis

$$\text{(H)} \qquad |f'(s)| \le C(1+|s|^\gamma),\ 0 \le \gamma \le \frac{2}{n-2} \text{ if } n \ge 3,\ 0 \le \gamma < \infty \text{ if } n = 2$$

$u_0(x) \in W^{2,p}(\Omega) \cap W^{2,2}(\Omega) \cap W_0^{1,p}(\Omega)$, then there exists a unique solution $u \in C([0,\infty); W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$, where $\max\left\{1, \frac{n}{2}\right\} < p < \infty$. Clearly the condition $\frac{n}{2} < p$, which is necessary if one uses the methods of [5-7], is very harsh. For example, according to this condition for the most important case $p = 2$ the values of n only can be $n \le 3$. So these results are no satisfactory. However up to now for the case $n \ge 2p$ the existence of global $W^{2,p}$ solution of the problem (1)-(3) is still open.

In this paper by using completely different method from [1-7] we study the problem (1)-(3) in arbitrary dimensions. We only assume that $\partial\Omega$ is sufficiently smooth, $f(s) \in C^1$ and satisfies (H), $u_0(x) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, then for any $T > 0$ we obtain a unique global solution $u \in W^{1,\infty}(0,T; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$, where $2 \le p < \infty$. So we have generalized and improved the known results essentially.

In this paper we always assume $\Omega \subset \mathbf{R}^n$ be a sufficiently smooth bounded domain, $\|\cdot\|_p$ denotes $L^p(\Omega)$ norm, $\|\cdot\| \equiv \|\cdot\|_2$, $\|\cdot\|_{k,p}$ denotes $W^{k,p}(\Omega)$ norm and $(u,v) = \int_\Omega u(x)v(x)dx$; C, C_i, M, M_i and E_i all denote the constants independent of u.

2. Global $W^{2,2}$ Solutions

Let $\{w_j(x)\}$ be a system of eigenfunctions of the problem $\Delta w_j + \lambda w_j = 0$ in Ω, $w_j\,|_{\partial\Omega} = 0$ construct approximate solutions of the problem (1)-(3) as follows

$$u_m(x,t) = \sum_{j=1}^{m} \alpha_{jm}(t) w_j(x), \quad m = 1, 2, \cdots \tag{4}$$

According to Galerkin method $\alpha_{jm}(t)$ satisfies

$$(u_{mt}, w_s) - (\Delta u_{mt}, w_s) = (\operatorname{div} f(u_m), w_s) \tag{5}$$

$$\alpha_{jm}(0) = a_{jm},\ s,\ j = 1, 2, \cdots, m \tag{6}$$

Lemma 1 *Assume that $f(s) \in C^1$, $u_0(x) \in W_0^{1,2}(\Omega)$, and choose a_{jm} such that $u_m(x,0) \xrightarrow{W^{1,2}} u_0(x)$, then we have*

$$\|u_m\|^2 + \|\nabla u_m\|^2 \equiv \|u_m(0)\|^2 + \|\nabla u_m(0)\|^2 \le E_1 \quad (0 \le t < \infty) \tag{7}$$

Proof Multiplying (5) by $\alpha_{sm}(t)$ and summing it for s we obtain

$$\frac{d}{dt}[\|u_m\|^2 + \|\nabla u_m\|^2] = -2(f(u_m), \operatorname{div} u_m)$$