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NON-EXISTENCE FOR HARMONIC MAPS ON COMPLETE NONCOMPACT MANIFOLDS

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Abstract In this paper, we prove a nonexistence theorem on harmonic maps. This generalizes the well-known Liouville-type theorem on harmonic maps due to S.Y. Cheng and H.I. Choi.

Key Words Asymptotically nonnegative curvature; harmonic map; convex ball non-existence.

Classification 58E20.

1. Introduction

Let M^m and N^n be two complete Riemannian manifolds. If their metrics are given locally by $ds^2_M = \sum_{i,j=1}^m g_{ij}dx^i dx^j$ and $ds^2_N = \sum_{\alpha,\beta=1}^n h_{\alpha\beta}du^\alpha du^\beta$, respectively, then the energy density function of a smooth map $u : M \to N$ is defined by

$$e(u)(x) = \sum_{i,j=1}^m \sum_{\alpha,\beta=1}^n g^{ij}(x)\frac{\partial u^\alpha}{\partial x^i}(x)\frac{\partial u^\beta}{\partial x^j}(x)h_{\alpha\beta}(u(x))$$

and the total energy of u is given by

$$E(u) = \int_M e(u)dx$$

Euler-Lagrangian equation for the critical points of the total energy functional can be written as

$$\Delta u^\alpha(x) = -\sum_{i,j=1}^m \sum_{\alpha,\beta=1}^n \Gamma^\alpha_{\beta\gamma}(u(x))g^{ij}(x)\frac{\partial u^\beta}{\partial x^i}(x)\frac{\partial u^\gamma}{\partial x^j}(x)$$

for all $1 \leq \alpha \leq m$, where $\{\Gamma^\alpha_{\beta\gamma}\}$ are the Christoffell symbols of N. The solutions of this equation are called harmonic maps. The study for the existence of harmonic maps between complete noncompact manifolds has been attracting many geometers and analysts [1]. In this note, we will study the non-existence of nonconstant harmonic maps from complete noncompact Riemannian manifolds with asymptotically nonnegative sectional curvature into other complete Riemannian manifolds, provided the image lies inside a geodesically convex ball.

Before stating our result, we first give some definitions. Let M^m be a complete noncompact Riemannian manifold. It has asymptotically nonnegative sectional curvature if for a fixed point $p \in M$, there exists a nonnegative continuous monotone nonincreasing function $\lambda : [0,\infty) \to [0,\infty)$ such that the integral $\int_0^\infty t\lambda(t)dt$ is finite and the sectional curvature of M at any point $x \in M$ is bounded from below by $-\lambda$ $(\mathrm{dist}_M(p,x))$. Clearly, for any fixed point $p \in M$, M is of the above property. Let D be a compact subset of M, then an unbounded component of $M\backslash D$ is called an end of M with respect to D. If the number of the ends of M with respect to any compact subset is uniformly bounded by an integer, then we say that M has finitely many ends, and the least upper bound is called the number of the ends of M. Let $B_Q(\tau)$ be a geodesically convex ball in a complete Riemannian manifold N^n, i.e., the geodesic ball centered at p with radius τ, $\tau < \frac{\pi}{2\sqrt{\kappa}}$, which lies inside the cut locus of Q, where κ is an upper bound of the sectional curvature of N, $\kappa \geq 0$. It is well-known that there exists a normal coordinate system around Q, which covers $B_Q(\tau)$. Now, we can state our result as follows:

Main Theorem *Let M be a complete noncompact Riemannian manifold with asymptotically nonnegative sectional curvature and only one end and $B_Q(\tau)$ be a geodesically convex ball. Then, any harmonic map $u : M \to B_Q(\tau)$ has to be constant.*

Remark 1) In this direction, the earliest result is due to S.T. Yau [2], who proved that there is no positive harmonic function on complete Riemannian manifolds with nonnegative Ricci curvature. This result was later generalized to the case of harmonic maps by Cheng [3] and then by Choi [4]: Any harmonic map has to be constant from complete Riemannian manifolds with nonnegative Ricci curvature into geodesically convex balls.

2) The existence theorem for harmonic maps in [5] tells us that the condition of "with only one end" connot be deleted.

In the following, we describe the proof of the main theorem. The proof consists of three ingrediends. The first one is the estimate for energy density of harmonic maps (Corollary 2.4) and Area comparison on geodesic spheres (Lemma 2.1). The gradient estimate is essentially due to Cheng [3] and Choi [4], we give it a precise version in the present case in order to use it in our proof conveniently. The second one is the behavior at infinity of bounded superharmonic functions (Lemma 2.2). The last one is to introduce some new measures on $M\backslash B_p(R)$ and $\partial B_p(R)$, which are related to the volume element on $M\backslash B_p(R)$ and the area element on $\partial B_p(R)$ respectively. Using the convex property of geodesically convex balls, we can construct some convex functions on them, which are in fact the square of the distance functions to certain points respectively. Then, the compositions of these convex functions and the harmonic map in question give rise to some bounded subharmonic functions. Using the new measures