J.Partial Diff. Eqs. 11(1998), 97–124
©International Academic Publishers Vol.11, No.2

VISCOUS BOUNDARY LAYERS AND THEIR STABILITY (I)*

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Dedicated to Professor Ding Xiaxi on the Occasion of His 70th Birthday

(Received July 23, 1997)

Abstract This paper is concerned with the asymptotic limiting behavior of solutions to one-dimensional quasilinear scalar viscous equations for small viscosity in the presence of boundaries. We consider only non-characteristic boundary conditions. The main goals are to understand the evolution of viscous boundary layers, to construct the leading asymptotic ansatz which is *uniformly valid* up to the boundaries, and to obtain rigorously the uniform convergence to smooth solution of the associated inviscid hyperbolic equations away from the boundaries.

Key Words Viscous boundary layer; nonlinear stability; Navier-Stokes equations; compressible waves; expansive wave; matched asymptotic analysis.

Classification 35L65, 35L35.

In this paper, we first construct formally the approximation solutions up to any order of accuracy by using the matched asymptotic analysis and multiple-scale expansions. Next, it is proved that for general viscous equations, *weak boundary layers* (those with suitably small strength) are always nonlinearly stable and the boundary layer effects are localized near the boundary so that the viscous solutions converge to the smooth inviscid solution uniformly away from the boundary. Furthermore, the asymptotic ansatz, constructed by the matched asymptotic analysis, can be justified rigorously up to any order. These results are true for general systems and for multiple space dimensions. This in particular implies the short time stability of viscous boundary layers. Finally, we show that strong boundary layers are nonlinearly stable also for viscous conservation laws with genuinely-nonlinear fluxes. The analysis for the last results depends crucially on the structure of the underling boundary layer. The proofs are based weighted energy estimates. The rate of convergence in viscosity is optimal.

* The research supported in part by NSF Grant DMS-9600137, and DOE Grant DO-FG02-88ER-25053.

1. Introduction and Main Results

The asymptotic equivalence between a viscous parabolic system such as compressible Navier-Stokes system and its associated inviscid hyperbolic equations such as compressible Euler equations in the limit of small dissipation is of considerable significance in many physical phenomena and their numerical computations. This is particularly so in the presence of shock discontinuities and boundaries (see [1]). The rigorous mathematical justification of this asymptotic equivalence poses challenging problems in many important cases due to the stiffness of the singular limit near boundaries and shock discontinuities, see the recent studies [2], [3], [4], and [5] and the references therein. The aim of this paper is to study such an asymptotic equivalence in the presence of boundaries. The goals are to understand the evolution and structure of viscous boundary layers and their interactions with interior inviscid hyperbolic flows, and to show the uniform convergence of the viscous solutions to the smooth inviscid flow away from the boundaries. To start with, we will concentrate on the model problem: one-dimensional scalar equations and smooth inviscid flows. We note that for scalar equations, with or without boundary, the convergence of viscous solutions in mean to even weak entropy solutions to the inviscid hyperbolic problem has been established by using maximum principle and entropy estimates [6, 7]. Yet, little information is given by this approach on the asymptotic behavior of the viscous solutions for small but non-zero viscosity. Furthermore, it seems to be extremely difficult to apply such an approach to any systems which do not admit weakly maximum principle [7]. Our analysis is based on matched multiple-scale expansions and weighted energy estimates, and can be, in principal, generalized to systems and multi-dimension. Indeed, some extensions of the results reported here to general systems in multi-dimension will be reported in a forthcoming paper (II).

Consider, the initial-boundary-value problem (IBVP)

$$\partial_t u^\varepsilon + \partial_x(f(u^\varepsilon, x, t)) + g(u^\varepsilon, x, t) = \varepsilon \partial_x^2 u^\varepsilon,$$

$$u^\varepsilon \in \mathbf{R}^1, \quad (x,t) \in \mathbf{R}^1_+ \times \mathbf{R}^1_+, \ \varepsilon > 0 \tag{1}$$

$$u^\varepsilon(x, t=0) = u_0(x) \tag{2}$$

$$u^\varepsilon(x=0, t) = u_1(t) \tag{3}$$

where f and g are smooth functions of its variables, and we require that

$$\partial_u f(u, x, t) < 0 \tag{4}$$

in the region in which we are interested, and the initial and boundary data are assumed to be smooth and compatible at the corner $x = t = 0$. We are seeking the