J.Partial Diff. Eqs. **11**(1998), 301-312
©International Academic Publishers Vol.11, No.4

SPACE-TIME ESTIMATES FOR PARABOLIC TYPE OPERATOR AND APPLICATION TO NONLINEAR PARABOLIC EQUATIONS *

Miao Changxing
(Centre for Nonlinear Studies, Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, China)

Gu Yonggeng
(Institute of Systems Science, Academia Sinica, Beijing 100080, China)

(Received Sept. 9, 1996; revised Jan. 20, 1997)

Abstract In this present paper we establish space-time estimates of solutions for linear parabolic type equations based on classical multipliers theory or operator semigroup theory. According to space-time estimates we first construct suitable work space $L^q(0,T;L^p)$, moreover we study the Cauchy problem and initial boundary value problem for semilinear parabolic equation in $L^q(0,T;L^p)$ type space.

Key Words Multipliers; parabolic type equations; Cauchy problem; initial boundary value problem; space-time estimate.

Classification 35K22, 35K25, 35K35.

1. Introduction

This paper is devoted to the study of Cauchy problem or initial boundary value (IBV) problem for semilinear parabolic equation

$$u_t + Au = F(u), \quad u(0) = \varphi(x) \in D(A) \tag{1.1}$$

where $A = -P_{2m}(D)$ denotes the elliptic operator on $L^p(\Omega)$, $\Omega = \mathbf{R}^n$ or $\Omega \subset \mathbf{R}^n$ is a bounded smooth domain. $D(A) = W^{2m,p}(\mathbf{R}^n)$ or $D(A) = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$. $P_{2m}(x)$ $(x \in \mathbf{R}^n)$ is $2m$-order polynomial with real part $\Re P_{2m} < 0$, $x \in \mathbf{R}^n \backslash \{0\}$, $F(u)$ denotes a nonlinear function such that

$$\begin{cases} |F(u) - F(v)| \le C(|u|^\alpha + |v|^\alpha)|u - v| \\ F(0) = 0 \end{cases} \tag{1.2}$$

* This project supported by the Post-doctoral Foundational of China and National Natural Science Foundation of China, No. 19601005.

A simple case of (1.1), (1.2) is semilinear heat equation

$$\begin{cases} u_t - \Delta u = |u|^\alpha u, & \alpha > 0,\ (t,x) \in [0,\infty) \times \Omega \\ u(0,x) = \varphi(x), & x \in \Omega \end{cases} \tag{1.3}$$

with boundary condition

$$u\,|_{\partial\Omega} = 0 \tag{1.4}$$

where Ω is a smooth bounded domain in $\mathbf{R}^n$ or $\Omega = \mathbf{R}^n$ itself; if $\Omega = \mathbf{R}^n$, we understand the problem (1.3), (1.4) as (1.3). Weissler [1] established the local solvability of (1.3) in $C([0,T); L^r(\mathbf{R}^n))$ for $\varphi(x) \in L^r(\mathbf{R}^n)$ if $r = \frac{n\alpha}{2} > 1$ and that T can be taken as infinity provided that $\|\varphi(x)\|_r$ is sufficiently small. Similar to wave equation and dispersive wave equation [2–4] we first show the space-time estimates for linear parabolic equation, and also construct suitable work-space $L^q(0,T;L^p)$ for nonlinear parabolic equation. Based on space-time estimates and Banach contraction principle we establish local solvability of Cauchy problem or IBV problem for semilinear parabolic equation in $L^q(0,T;L^p)$ with $\varphi(x) \in L^r$. Moreover we also prove that T can be taken as infinity provided that $\|\varphi(x)\|_r$ is sufficiently small, where (p,q,r) is an admissible triple (See the latter section).

Our plan in this present paper is as following: In Section 2 we propose the reduction of problem and state main results. In Section 3 we introduce the concept of admissible triple (p,q,r) and also obtain space-time estimates and some basic estimates for linear parabolic equation based on multipliers theory. Section 4 is devoted to the proof of the main results.

We conclude this section with several notations given. $\Re f(x)$ denotes the real part of $f(x)$, ω_{n-1} denotes the volume of unit sphere in $\mathbf{R}^n$. For $1 < p \le \infty$, let $p' = \frac{p}{p-1}$ be the dual exponent of p, $L^p(\Omega)$ denotes standard Lebesgue space with norm $\|\cdot\|_p$, $W^{m,p}(\Omega)$ $(m \in N \cup \{0\})$ is usual Sobolev space with norm $\|\cdot\|_{m,p} = \sum_{|\alpha|\le m} \|\partial^\alpha \cdot\|_p$, where α denotes n-th index. $\dot{W}^{m,p}(\Omega)$ denotes homogeneous Sobolev space respect to $W^{m,p}(\Omega)$ with norm $\|\cdot\|_{\dot{m},p} = \sum_{|\alpha|=m} \|\partial^\alpha \cdot\|_p$. For any $v \in \mathcal{S}(\mathbf{R}^n)$, $\mathcal{F}v$ and $\mathcal{F}^{-1}v$ denote the Fourier transform and Fourier inverse transform of v in $\mathbf{R}^n$ respectively. For $r \in \mathbf{R}$, $H^{r,p}$ and $\dot{H}^{r,p}$ $(1 < p < \infty)$ denote Bessel potential space with norm

$$\|\cdot\|_{H^{r,p}} = \|(I-\Delta)^{\frac{r}{2}} \cdot\|_p = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{r}{2}}\mathcal{F}\cdot]\|_p$$

and Riesz potential space norm

$$\|\cdot\|_{\dot{H}^{r,p}} = \|(-\Delta)^{\frac{r}{2}} \cdot\|_p = \|\mathcal{F}^{-1}[|\xi|^r \mathcal{F}\cdot]\|_p$$