J.Partial Diff. Eqs. **11**(1998), 335–346
©International Academic Publishers Vol.11, No.4

EXISTENCE OF PHYSICAL SOLUTIONS TO THE BOUNDARY VALUE PROBLEM OF AN EQUATION IN MODELING OIL FILM SPREADING OVER A SOLID SURFACE

Gao Wenjie and Yin Jingxue
(Department of Mathematics, Jilin Univ., Changchun 130023, China)
(Received Nov. 26, 1996; revised May 18, 1997)

Abstract A boundary value problem to a fourth order nonlinear degenerate parabolic equation arising in modeling oil film spreading over a solid surface is studied in the present paper. Basing on the consideration of physical phenomenon described by the original model, the authors forcus mainly on the case of two-dimensional space. The existence of nonnegative and radial solutions is established for nonnegative initial datum.
Key Words Existence; physical solution; oil film.
Classification 35K55, 35K65, 35Q35, 76E30.

1. Introduction

In the present paper, we study the partial differential equation

$$\frac{\partial u}{\partial t} + \nabla(f(u)\nabla\Delta u) = 0 \tag{1.1}$$

with the following initial and boundary value conditions

$$\frac{\partial u}{\partial n}\Big|_{\partial B} = \frac{\partial \Delta u}{\partial n}\Big|_{\partial B} = 0 \tag{1.2}$$

and

$$u\,|_{t=0}= u_0(x) \tag{1.3}$$

where $f(u)$ is a nonnegative function and B the unit ball in the plane region.

The equation (1.1) arises in modeling the motion of oil film spreading over a solid surface. Many authors have studied the equation (1.1) for space dimension 1 (See, for instance, the survey paper [1] and the references therein). F. Bernis and A. Friedman [2] proved the existence of weak solutions to the problem and studied some other properties of the solutions such as the nonnegativity of the solutions with nonnegative initial data and the increasing property of the supporting set of the solutions. There are also some other works relating to the above problem, see [3–5], [6–10] and the references therein for details.

Notice that the actual model for the motion of oil film spreading over a solid surface occurs in 2-dimensional space. To describe the motion, we should consider the problem in higher dimensional spaces.

For simplicity, we study the radial solution of the problem (1.1)–(1.3). We will study the problem in space dimension 2 because of its physical background. It is easy to verify that a radial solution of the problem satisfies

$$\begin{cases} \dfrac{\partial(ru)}{\partial t}+\dfrac{\partial}{\partial r}\Big(rf(u)\dfrac{\partial V}{\partial r}\Big)=0 \\ rV=\dfrac{\partial}{\partial r}\Big(r\dfrac{\partial u}{\partial r}\Big) \\ \dfrac{\partial u}{\partial r}\Big|_{r=0}=\dfrac{\partial u}{\partial r}\Big|_{r=1}=\dfrac{\partial V}{\partial r}\Big|_{r=0}=\dfrac{\partial V}{\partial r}\Big|_{r=1}=0 \\ u\,|_{t=0}=u_0(r) \end{cases} \tag{1.4}$$

One may notice that the main difference of the 2-D problem from the 1-D problem is that the equation is degenerate at $r=0$ and hence the arguments for 1-D problem can not be applied directly. We will put our attention on overcoming the difficulty arising from the degeneracy at $r=0$ and mainly prove the existence of a weak solution in the sense of the following.

Definition *A function u is said to be a weak solution of the problem* (1.4) *if the following conditions are fulfilled:*

(1) *$ru(r,t)$ is continuous in $\overline{Q}_T$ and $u_t, u_r, u_{rr}, u_{rrr}$ and u_{rrrr} are all in $C(P)$, where $P=\overline{Q}_T\backslash(\{u=0\}\cup\{t=0\}\cup\{r=0\})$;*

(2) *$\sqrt{rf(u)}u_{rrr}\in L^2(P)$;*

(3) *For any $\phi\in \mathrm{Lip}\,(\overline{Q}_T)$, $\phi=0$ near $t=0$ and $t=T$, the following integral equality holds:*

$$\iint_{Q_T} ru\phi_t dtdx+\iint_P rf(u)V_r\phi_r dtdx=0 \tag{1.5}$$

(4)

$$u(x,0)=u_0(x),\quad x\in[0,1] \tag{1.6}$$

$$ru_r(\cdot,t)\to u_{0r},\quad \text{strongly in } L^2(I) \text{ as } t\to 0 \tag{1.7}$$

and

$$u \text{ satisfies the lateral boundary value condition where } u\neq 0. \tag{1.8}$$