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DETERMINATION OF A PARAMETER $h(t)$ FOR A PHASE FIELD MODEL

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Abstract A phase field model with an unknown parameter $h(t)$ is considered. The existence, uniqueness and continuous dependence upon the data of the solution (u, φ, h) are demonstrated.

Key Words Unknown parameter; inverse problem; phase field model.
Classification 35Q99, 35R30.

1. Introduction

In this paper we will consider the following inverse problem: we wish to find the evolution parameter $h(t)$ and the functions $u(x,t), \varphi(x,t)$, such that

$$u_t + k\varphi_t = \Delta u + h(t)\overline{f}(x,t), \qquad (x,t) \in Q \tag{1.1}$$

$$\varphi_t = \Delta\varphi + a(x,t)\varphi + b(x,t)\varphi^2 - \varphi^3 + u + g(x,t), \quad (x,t) \in Q \tag{1.2}$$

$$u(x,0) = u_0(x), \quad \varphi(x,0) = \varphi_0(x), \qquad x \in \Omega \tag{1.3}$$

$$u = \varphi = 0, \qquad (x,t) \in S \tag{1.4}$$

$$\int_\Omega H(x,t)u(x,t)dx = E(t) \qquad 0 \le t \le T \tag{1.5}$$

where $Q = \Omega \times (0,T]$, $S = \partial\Omega \times (0,T]$, and Ω is an open bounded domain in $\mathbf{R}^3$ with boundary $\partial\Omega \in C^2$. $T > 0$ and $k > 0$ are given constants, $a, b, \overline{f}, g, u_0, \varphi_0, H$, and E are given functions.

It is well known that G. Caginalp proposed the system of phase field equations in [1]. The phase field model represents a refinement of the classical Stefan model for the transition between the solid and liquid phases of a material. The authors of [1], [2] proved the global existence of solutions to the phase field equations with Dirichlet or

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Neumann boundary conditions. The optimal control problem for a phase field model has been studied in [3].

The problem (1.1)—(1.5) can be regarded as a control problem with source control for a phase field model. Here we study the identification of the control $h(t)$ necessary to produce the specified or desired energy $E(t)$. The aim of this article is to investigate the existence, uniqueness and dependence upon the data of the solution $(u, \varphi, h) \in W_p^{2,1}(Q) \times W_q^{2,1}(Q) \times L_p(0,T)$ for the problem (1.1)—(1.5), where p, q are given constants and satisfy

$$p \geq 2$$

and

$$p \leq q \leq p^* = \frac{5p}{5-2p} \quad \text{for } p \in \left[2, \frac{5}{2}\right)$$

$$p \leq q < +\infty \quad \text{for } p \in \left[\frac{5}{2}, +\infty\right)$$

It is convenient here to list our assumptions on data.

(H1) $a(x,t), b(x,t) \in L_\infty(Q)$,

(H2) $\overline{f}(x,t) \in L_\infty(0,T; L_p(\Omega))$, $g(x,t) \in L_q(Q)$,

(H3) $u_0(x) \in W_p^{2-2/p}(\Omega) \cap \overset{\circ}{W}{}_p^1(\Omega)$, $\varphi_0(x) \in W_q^{2-2/q}(\Omega) \cap \overset{\circ}{W}{}_q^1(\Omega)$,

(H4) $H(x,t) \in C^{1,1}(\overline{Q})$, $\int_\Omega H(x,t)\overline{f}(x,t)dx \geq \sigma_0$, and constant $\sigma_0 > 0$,

(H5) $E(t) \in W_p^1(0,T)$ and $E(0) = \int_\Omega H(x,0)u_0(x)dx$.

Throughout the standard space notations, such as $C^{k,l}(\overline{Q}), L_p(Q), W_p^{k,l}(Q), W_p^k(\Omega)$, $\overset{\circ}{W}{}_p^k(\Omega), L_p(0,T)$, and $W_p^l(0,T)$ (see [4]) are used, and the space

$$L_\infty(0,T; L_p(\Omega)) = \{f \mid \|f(\cdot,t)\|_{L_p(\Omega)} \in L_\infty(0,T)\}$$

Now let $(u, \varphi, h) \in W_p^{2,1}(Q) \times W_q^{2,1}(Q) \times L_p(0,T)$ be a solution for (1.1)—(1.5). It is not difficult to show that our assumptions imply

$$\begin{aligned} E'(t) &= \int_\Omega (H_t u + H u_t)dx \\ &= \int_\Omega (H_t u - \nabla H \nabla u - kH\varphi_t)dx + \int_{\partial\Omega} H\frac{\partial u}{\partial n}d\sigma + h(t)\int_\Omega H\overline{f}dx \end{aligned} \quad (1.6)$$

where $\frac{\partial u}{\partial n}$ is the outer normal derivative, and

$$\nabla u = (u_{x_1}, u_{x_2}, u_{x_3}), \quad \nabla H = (H_{x_1}, H_{x_2}, H_{x_3})$$

On the assumption that (H4) allows us to solve for $h(t) \in L_p(0,T)$ in (1.6) explicitly

$$h(t) = \left(\int_\Omega H\overline{f}dx\right)^{-1}\left(E'(t) + \int_\Omega (kH\varphi_t - uH_t + \nabla H \nabla u)dx - \int_{\partial\Omega} H\frac{\partial u}{\partial n}d\sigma\right) \quad (1.7)$$