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PINNING OF VORTICES FOR A VARIATIONAL PROBLEM RELATED TO THE SUPERCONDUCTING THIN FILMS HAVING VARIABLE THICKNESS*

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Abstract This paper is concerned with the minimization problem related to the superconducting thin films having variable thickness. The asymptotic behavior of the minimizers is discussed. The singularities are found to be located at the thinnest positions of the film and the vortices of u_ε are proved to be pinned near these regions.

Key Words Superconductivity; vortices; asymptotic behavior; pinning mechanism.

Classification 35J55, 35Q40.

1. Introduction

Three-dimensional thin films of superconducting material, say $\Omega \times (-\delta a(x), \delta a(x))$, are modeled as two-dimensional objects by Q. Du and M.D. Gunzburger in [1]. The reduced model is derived in [1] (see (1.1) in the following). It is believed from numerical computation that vortices are pinned near the relatively thin regions of the sample. The contexts of our study are the asymptotic behavior of the minimizers and the pinning mechanism of vortices.

Scaling the physical parameters in the model derived in [1], we consider the following functional

$$G_\varepsilon(u, A) = \frac{1}{2}\int_\Omega a(x)\Big\{|\nabla_A u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 + |dA|^2\Big\} \tag{1.1}$$

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where Ω is a bounded smooth domain in $\mathbf{R}^2$, $a(x)$ is smooth such that $0 < a_0^{-1} \le a(x) \le a_0$, $|\nabla a|$, $|D^2 a| \le a_0$ in $\overline{\Omega}$ with constant $a_0 > 0$, $u \in H^1(\Omega, \mathbf{R}^2)$ and A, the vector potential, is a real valued 1-form: $A = A_1 dx_1 + A_2 dx_2$, $\nabla_A u = \nabla u - iAu$.

As it is well known, the main characteristic of $G_\varepsilon(u, A)$ is its gauge invariance, i.e., $G_\varepsilon(u_\phi, a_\phi) = G_\varepsilon(u, A)$ if $\phi \in H^2(\Omega, \mathbf{R})$ and $u_\phi = ue^{i\phi}$, $A_\phi = A + d\phi$. In this case, we say that (u, A) is gauge equivalent to (u_ϕ, A_ϕ).

Since a Dirichlet-type condition is not consistent with the gauge invariance, we proceed instead as follows to have a well-posed minimization problem. Let $d > 0$ be an integer and $g : \partial\Omega \to \mathbf{R}$ be a smooth function.

Consider the space (see [2])

$$V = \{(u, A) \in H^1(\Omega, \mathbf{R}^2) \times H^1(\Omega, \mathbf{R}^2) :$$
$$|u| = 1 \text{ on } \partial\Omega, \deg(u, \partial\Omega) = d > 0, J \cdot \tau = g \text{ on } \partial\Omega\}$$

where τ denotes the unit tangent vector to $\partial\Omega$ such that (ν, τ) is a direct, ν denotes the exterior normal to $\partial\Omega$, and $J = (iu, \nabla_A u)$ where $(a, b) = \frac{1}{2}(a\bar{b} + \bar{a}b)$ for complex numbers a, b. It is clear from [2] that $\inf\limits_{(u,A)\in V} G_\varepsilon(u, A)$ is achieved.

In the sequel, we choose the gauge transformation as follows ([1]): $d^*(a(x)A) = 0$ in Ω, $A \cdot \nu = 0$ on $\partial\Omega$. From [1], we know, there is a minimizer $(u_\varepsilon, A_\varepsilon)$ of (1.1) in V such that $d^*(a(x)A_\varepsilon) = 0$ in Ω, $A_\varepsilon \cdot \nu = 0$ on $\partial\Omega$.

Similarly to [2], we know, for $(u, A) \in H^1 \times H^1$, there is $(\tilde{u}, \tilde{A}) \in H^1 \times H^1$ gauge equivalent to (u, A) such that $d^*(a(x)\tilde{A}) = 0$ in Ω, $\tilde{A} \cdot \nu = 0$ on $\partial\Omega$. This claim has a local version, i.e., Ω can be replaced by G, any smooth subset of Ω. In what follows, for the minimizer of G_ε in V, we always assume that it is subject to above gauge transformation.

Carrying out an asymptotic analysis, we are able to prove the following theorems, our main results. For this aim, we distinguish two cases. Let $m = \min\limits_{\overline{\Omega}} a(x)$ and $a^{-1}(m) = \{x \in \overline{\Omega} \mid a(x) = m\}$.

Case I $a^{-1}(m) \subset \Omega$:

$$N = \operatorname{Card} a^{-1}(m) \ge d \tag{I(i)}$$

or

$$N = \operatorname{Card} a^{-1}(m) < d \tag{I(ii)}$$

Case II $a^{-1}(m) \cap \partial\Omega \ne \emptyset$:

$$N = \operatorname{Card}(a^{-1}(m) \cap \Omega) \ge d \tag{II(i)}$$