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# INITIAL VALUE PROBLEM FOR GENERAL QUASILINEAR HYPERBOLIC SYSTEMS WITH CHARACTERISTICS WITH CONSTANT MULTIPLICITY

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**Dedicated to Professor Ding Xiaxi on the Occasion of His 70th Birthday**

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**Abstract** The authors consider the global existence and the blow-up phenomenon of classical solutions with small amplitude to the Cauchy problem for general quasilinear hyperbolic systems with characteristics with constant multiplicity and given some applications.

**Key Words** Global classical solution; general quasilinear hyperbolic system; characteristic with constant multiplicity.

**Classification** 35L45, 35L67.

## 1. Introduction

Consider the first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0 \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unkown vector function of $(t, x)$ and $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with $C^2$ elements $a_{ij}(u)$ $(i, j = 1, \cdots, n)$.

By hyperbolicity, for any given $u$ on the domain under consideration, $A(u)$ has $n$ real eigenvalues $\lambda_1(u), \cdots, \lambda_n(u)$ and a complete system of left (resp. right) eigenvectors. For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)) \tag{1.2}$$

We have

$$\det|l_{ij}(u)| \neq 0 \quad (\text{equivalently, } \det|r_{ij}(u)| \neq 0) \tag{1.3}$$

All $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i,j=1,\cdots,n)$ are supposed to have the same regularity as $a_{ij}(u)$ $(i,j=1,\cdots,n)$.

Without loss of generality, we suppose that on the domain under consideration

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i,j=1,\cdots,n) \tag{1.4}$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i=1,\cdots,n) \tag{1.5}$$

where $\delta_{ij}$ stands for the Kronecker's symbol.

In this paper, we suppose that on the domain under consideration, each eigenvalue of $A(u)$ has a constant multiplicity. Without loss of generality, we may suppose that

$$\lambda(u) \triangleq \lambda_1(u) \equiv \cdots \equiv \lambda_p(u) < \lambda_{p+1}(u) < \cdots < \lambda_n(u) \tag{1.6}$$

where $1 \le p \le n$. When $p=1$, the system (1.1) is strictly hyperbolic; while, when $p>1$, (1.1) is a non-strictly hyperbolic system with characteristics with constant multiplicity.

We suppose furthermore that on the domain under consideration, each multiple characteristic is *linearly degenerate* in the sense of P.D. Lax. Then, by (1.6), when $p>1$ we have

$$\nabla\lambda(u)r_i(u) \equiv 0 \quad (i=1,\cdots,p). \tag{1.7}$$

**Remark 1.1** If the system (1.1) can be written in the form of conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \tag{1.8}$$

where $f(u) = (f_1(u),\cdots,f_n(u))^T$, then (1.6) implies (1.7) (See G. Boillat [1] and H. Freistühler [2]).

For the initial data

$$t=0: u=\phi(x) \tag{1.9}$$

where $\phi(x)$ is a "small" $C^1$ vector function of $x$ with certain decay properties as $|x| \to +\infty$, we shall investigate the global existence or the blow-up phenomenon of $C^1$ solution to the Cauchy problem (1.1) and (1.9).

We point out that in the strictly hyperbolic case:

$$\lambda_1(u) < \lambda_2(u) \cdots < \lambda_n(u) \tag{1.10}$$

suppose that there exists a nonempty set $J \subseteq \{1,2,\cdots,n\}$ such that if $i \in J$, then $\lambda_i(u)$ is *genuinely nonlinear* in the sense of P.D. Lax:

$$\nabla\lambda_i(u)r_i(u) \neq 0 \tag{1.11}$$

while, if $i \overline{\in} J$, then $\lambda_i(u)$ is linearly degenerate in the sense of P.D. Lax:

$$\nabla\lambda(u)r_i(u) \equiv 0 \tag{1.12}$$