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REGULARIZATION OF SOLUTION OPERATORS FOR QUASILINEAR HYPERBOLIC-PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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Dedicated to Professor Ding Xiaxi on the Occasion of His 70th Birthday

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Abstract We list a hierarchy of hyperbolic-parabolic partial differential equations in terms of the regularization properties of their solution operators. This ranges from the most regularizing of the heat operator to the least, that of the hyperbolic conservation laws. We illustrate this with physical examples in gas dynamics and mechanics.

Key Words Nonlinear hyperbolicity; dissipations; conservation laws; regularity of solutions.

Classification 35F20, 35G05.

1. Introduction

It is well-known that the solution $u(x,t)$, $t > 0$, of the heat equation

$$u_t = u_{xx}$$

is smooth even if the initial value $u(x,0)$ is rough; while those of the hyperbolic conservation laws

$$u_t + f(u)_x = 0$$

in general contains discontinuous shock waves even if the initial data are smooth. The linear wave equation

$$u_t + cu_x = 0$$

is in between: its solutions maintain the same degree of regularity of the initial values. There are several physical systems whose solution operators have different degrees of regularizing properties. By regularization we take a more general view than the local in time regularizing properties as is usual with the study of linear partial differential equations. In particular, we are interested in the large-time behavior. For instance, the hyperbolic conservation laws may be viewed as being more regular than the linear wave equation because the nonlinearity of the flux $f(u)$ may damp the oscillation of the initial data; while the linear wave equation carries the initial oscillation to later time. In other words, there should not be a linear order of the hierarchy of the regularization. We illustrate these with physical examples in the following sections.

2. Linear Heat Equation

The initial value problem for the linear heat equation

$$u_t = u_{xx} \tag{2.1}$$

is solved by the convolution with the heat kernel with the initial value

$$\int_{-\infty}^{\infty} G(x-y,t)u(y,0)dy, \quad G(x,t) \equiv \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}} \tag{2.2}$$

The solution $u(x,t)$, $t > 0$, is C^∞ for any initial value which is mearly integrable. Moreover, it dissipates at the rate of $t^{1/2}$ in its essential support. This is evident from the inspection of the heat kernel: given any $\varepsilon > 0$ and $C > 0$,

$$G(x,t) = \begin{cases} O(1)t^{-\frac{1}{2}}, & x \le Ct^{\frac{1}{2}} \\ O(1)e^{-Dt}, & x \ge t^{\frac{1}{2}+\varepsilon} \end{cases} \tag{2.3}$$

for some positive D. We will discuss later that the same degree of regularization is present in not only hyperbolic-parabolic, but also purely hyperbolic systems.

3. Nonlinear Heat Equation

The simplest strong nonlinearity to add to the heat equation is to have convection with quadratic nonlinearity, the Burgers equation:

$$u_t + \left(\frac{u^2}{2}\right)_x = u_{xx} \tag{3.1}$$

This basic equation has certain scaling property and can be solved explicitly by transforming it into the heat equation through the Hopf-Cole transformation:

$$u \equiv -2\frac{w_x}{w} \tag{3.2}$$

$$w_t = w_{xx} \tag{3.3}$$