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# PROPERTIES ON INTERFACE OF SOLUTIONS FOR A DOUBLY DEGENERATE PARABOLIC EQUATION*

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**Abstract** We investigate the properties of the interface of the solution for a doubly nonlinear degenerate parabolic equation and show that the interface is nondecreasing and Lipschitz continuous.
**Key Words** Degeneracy; parabolic equation; interface; Lipschitz continuity.
**Classification** 35K.

## 1. Introduction

Kalashnikov [1] has derived a sufficient condition

$$\int_0^1 \frac{\phi'(s)}{\psi^{-1}(\sigma s)} ds < +\infty, \quad \forall \sigma > 0 \tag{1.1}$$

under which the solution of the following Cauchy problem for the doubly nonlinear degenerate parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\psi\Big(\frac{\partial}{\partial x}\phi(u)\Big), \quad (t,x) \in Q = (0,+\infty) \times R \tag{1.2}$$

has the property of finite speed of propagation of perturbations (FSPP). In other words, there exists an interface between the portion $\{u > 0\}$ and $\{u = 0\}$.

The special case of the equation (1.2) is the porous medium equation ($\psi'(\sigma) = 1$), and an extensive literature has devoted to the properties of the interface, see [2], [3] for a survey. Another typical example of the equation (1.2) is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(\Big|\frac{\partial u^m}{\partial x}\Big|^{p-1}\frac{\partial u^m}{\partial x}\Big) \tag{1.3}$$

which arises from the theory of non-Newtonian fluids. Esteban & Vazquez [4] showed that the interface of the equation (1.3) is nondecreasing and Lipschitz continuous. However, as far as we know, the results related to the general equation (1.2) are quite

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fragmentary. This is due mainly to the inherent difficulty caused by double nonlinearity and degeneracy. In this note, we investigate the properties of the interface of solutions for the equation (1.2). Our discussion is based on the comparison between solutions considered and super-(sub-) solutions. Indeed, the comparison result permits us to make use of different kind "travelling-wave" (super-, sub-)solutions in deducing the monotonicity and Lipschitz continuity of the interface.

## 2. The Main Result and Its Proof

To discuss the properties of the interface, it should be usually assumed that some smoothness conditions imposed on the known functions $\psi, \phi$ and the initial value $u_0(x)$ hold. However, we do not want to minimize the conditions here. In what follows, we simply assume that $\psi, \phi$ and $u_0(x)$ are appropriately smooth.

As for the structure, we need the following

(H1) $\psi(-s) = -\psi(s)$, $\psi'(s) > 0$ for $s \neq 0$ and $\lim_{s\to+\infty} \psi(s) = +\infty$,

(H2) $\phi(0) = 0$, $\phi'(s) > 0$ for $s > 0$.

In addition, since we are now treating the properties of the interface, we consider only nonnegative solutions and assume that (1.1) is always valid.

**Definition** *A function $u \in C(Q_T) \cap BV(Q_T)$ is said to be a continuous BV solution (supersolution, subsolution) of the equation* (1.2), *if the generalized derivative* $\dfrac{\partial\phi(u)}{\partial x} \in L^\infty(Q_T)$ *and for any testing function* $(0 \le)\xi \in C_0^\infty(Q_T)$

$$Q[u,\xi] \equiv \iint_{Q_T} u\frac{\partial \xi}{\partial t}dxdt - \iint_{Q_T} \psi\Big(\frac{\partial}{\partial x}\phi(u)\Big)\frac{\partial \xi}{\partial x}dxdt = (\le, \ge)0 \tag{2.1}$$

*where* $Q_T = \{(t,x); 0 < t \le T, x \in \mathbf{R}\}$.

As for the existence and uniqueness of $BV$ solutions, we refer to [5]. We note that the continuity of solutions is ensured by the strict monotonicity of $\phi$ and some smoothness conditions on the known functions.

We are now in the position to state the following theorem which plays an important role in our discussion to the properties of interfaces.

**Theorem 2.1**(Comparison Principle) *Let $u_1, u_2$ be a subsolution and a supersolution of the equation* (1.2) *with boundary value $u_1(t,a_1) \le u_2(t,a_1)$, $u_1(t,a_2) \le u_2(t,a_2)$ and initial value $u_1(0,x) \le u_2(0,x)$ where $a_1 < a_2$. Then $u_1(t,x) \le u_2(t,x)$.*

Because the main idea of the proof follows basically from the one given in [6], we omit the details here.

To discuss the properties of the interfaces, we first give the estimates of the velocity of propagation of disturbances with respect to the level $\{u = 0\}$, defined by

$$V(t,x) = -\frac{\psi(\phi(u)_x)}{u}, \quad \text{if } u > 0$$