INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 10, Number 2, Pages 298–313



# THE IMMERSED FINITE ELEMENT METHOD FOR PARABOLIC PROBLEMS USING THE LAPLACE TRANSFORMATION IN TIME DISCRETIZATION

TAO LIN AND DONGWOO SHEEN

**Abstract.** In this paper we are interested in solving parabolic problems with a piecewise constant diffusion coefficient on structured Cartesian meshes. The aim of this paper is to investigate the applicability and convergence behavior of combining two non-conventional but innovative methods: the Laplace transformation method in the discretization of the time variable and the immerse finite element method (IFEM) in the discretization of the space variable. The Laplace transformation in time leads to a set of Helmholtz-like problems independent of each other, which can be solved in highly parallel. The employment of immerse finite elements (IFEs) makes it possible to use a structured mesh, such as a simple Cartesian mesh, for the discretization of the space variable even if the material interface (across which the diffusion coefficient is discontinuous) is non-trivial. Numerical examples presented indicate that the combination of these two methods can perform optimally from the point of view of the degrees of polynomial spaces employed in the IFE spaces.

**Key words.** Immersed finite element method, interface problems, Laplace transform, parallel algorithm

## 1. Introduction

We consider solving the following parabolic interface problem:

$$\frac{\partial u}{\partial t} - \nabla \cdot a(\mathbf{x})\nabla u = f(\mathbf{x},t), \quad (\mathbf{x},t) \in \Omega \times (0,T), \tag{1.1a}$$

$$u(\mathbf{x},t) = g(\mathbf{x},t), \quad (\mathbf{x},t) \in \partial\Omega \times (0,T), \tag{1.1b}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{1.1c}$$

where the domain $\Omega$ is decomposed into two subdomains $\Omega^+$ and $\Omega^-$ by an interface $\gamma$ such that $\overline{\Omega} = \overline{\Omega^+} \cup \overline{\Omega^-}$ with $\Omega^+ \cap \Omega^- = \emptyset$. In (1.1), the diffusion coefficient $a(\mathbf{x})$ is a piecewise constant function such that

$$a(\mathbf{x}) = \begin{cases} a^- > 0, & \mathbf{x} \in \Omega^-, \\ a^+ > 0, & \mathbf{x} \in \Omega^+, \end{cases}$$

with $a^- \neq a^+$ in general; see Figure 1 for an illustration.

The discontinuity in the coefficient $a(\mathbf{x})$ leads to the following interface jump conditions to be satisfied by $u(\mathbf{x},t)$:

$$[u]_\gamma = 0 \quad \text{and} \quad \left[a\frac{\partial u}{\partial \mathbf{n}}\right]_\gamma = 0, \tag{1.2}$$

where $\mathbf{n}$ is the unit normal on the interface $\gamma$ towards $\Omega^+$.

In this paper, we apply two non-conventional but innovative approaches in solving (1.1). The time discretization is accomplished by the Laplace transformation method and the space discretization is carried out through the immersed finite

Received by the editors April 8, 2010 and, in revised form, January 4, 2012.
2000 *Mathematics Subject Classification.* 65N30, 44A10, 35K20.
This research work is partially supported by NSF grant DMS-0713763, DMS-1016313, NRF-2008-C00043 and NRF-2009-0080533.

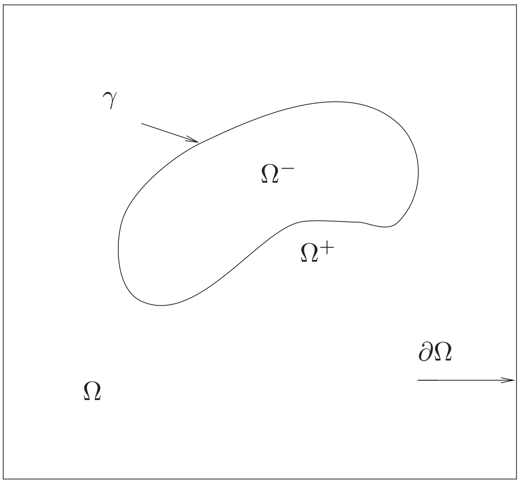


FIGURE 1. The domain $\Omega$ of the interface problem is separated by the interface $\gamma$ across which the coefficient has a jump discontinuity.

element methods (IFEMs). The Laplace transformation approach allows us to generate an approximate solution to the parabolic initial boundary value problem (IBVP) through the solutions to a set of independent Helmholtz-like interface problems. In addition, IFEMs allow us to use interface independent meshes, such as structured Cartesian meshes if preferred, to solve parabolic IBVPs whose diffusion coefficients are discontinuous.

Instead of solving the parabolic problem (1.1) by using traditional time-marching algorithms, such as the backward Euler scheme and the Crank-Nicolson scheme, we apply the Laplace transformation method for the discretization in the time direction. For each $z$ on a suitable contour $\Gamma \subset \mathbb{C}$, we denote by $\widehat{u}(\mathbf{x}, z)$ the standard Laplace transform in time of a function $u(\mathbf{x}, t)$:

$$\widehat{u}(\cdot, z) := \mathcal{L}[u](z) = \int_0^\infty u(\cdot, t)e^{-zt}\, dt. \tag{1.3}$$

The Laplace transforms of (1.1) are then given in the form

$$\begin{aligned} z\widehat{u}(\mathbf{x}, z) - \nabla \cdot\, a(\mathbf{x})\nabla\, \widehat{u}(\mathbf{x}, z) &= u_0(\mathbf{x}) + \widehat{f}(\mathbf{x}, z), \\ &\qquad (\mathbf{x}, z) \in \Omega \times \Gamma, \qquad (1.4a) \\ \widehat{u}(\mathbf{x}, z) &= \widehat{g}(\mathbf{x}, z), \\ &\qquad (\mathbf{x}, z) \in \partial\Omega \times \Gamma. \qquad (1.4b) \end{aligned}$$

For each $z \in \Gamma$, let $\mathcal{S}(z) : L^2(\Omega) \times H^{1/2}(\partial\Omega) \to H^1(\Omega)$ be the solution operator associated with the above complex-valued Helmholtz-like problem (1.4) so that

$$\widehat{u}(\cdot, z) = \mathcal{S}(z)\left(u_0(\cdot) + \widehat{f}(\cdot, z), \widehat{g}(\cdot, z)\right). \tag{1.5}$$

By the *Laplace inversion formula* ([3]), the time-domain solution to the parabolic interface problem (1.1) is given by

$$\begin{aligned} u(\cdot, t) &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \widehat{u}(\cdot, z)e^{zt}\, dz & (1.6a) \\ &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathcal{S}(z)\left(u_0(\cdot) + \widehat{f}(\cdot, z), \widehat{g}(\cdot, z)\right) e^{zt}\, dz. & (1.6b) \end{aligned}$$