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GLOBAL STABILITY FOR THE CAUCHY PROBLEM OF A CLASS OF REACTION-DIFFUSION SYSTEMS*

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Abstract By using invariant region argument together with Liapunov function technique for ODE, we give under certain circumstances a global stability analysis of the solutions for Cauchy problem of reaction-diffusion systems.
Key Words Reaction-diffusion systems; Cauchy problem; global stability.
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1. Introduction

Consider following reaction-diffusion system

$$(u_i)_t - D_i\Delta u_i = f_i(u_1, \cdots, u_n), \quad (t, x) \in [0, \infty) \times R^m_+$$

where $D_i > 0 (i = 1, \cdots, n)$ are diffusion constants, $f_i(u_1, \cdots, u_n)$ smooth functions on R^n_+. If we denote $u = (u_1, \cdots, u_n)$, $f(u) = (f_1(u), \cdots, f_n(u))$ and $D = \text{diag}\,(D_1, \cdots, D_n)$, the above system takes the vector form

$$u_t = D\Delta u + f(u), \quad (t, x) \in [0, \infty) \times R^m_+ \tag{1}$$

In the following discussion, we assume that system (1) possesses a unique positive equilibrium solution $u^* = (u_1^*, \cdots, u_n^*)$, i.e., $f(u^*) = 0$, $u^* > 0$.

Corresponding to system (1), the following Cauchy condition will be considered in this note:

$$u(0, x) = u^0(x), \quad x \in R^m \tag{2}$$

where $u^0(x)$ is smooth and the derivative of $u^0(x)$ is uniformly bounded and there are two constant vectors C_1 and $C_2 (C_1 \geq C_2 > 0)$ such that

$$C_2 \leq u^0(x) \leq C_1$$

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The smoothness of $f(u)$ and $u^0(x)$ ensures the existence and uniqueness of local solution for the problem (1) and (2). A proof of this conclusion follows from the fundamental solution argument in [1] (Th. 14.2).

The purpose of the present paper is to extend the proving method of [2] which discussed the global stability of positive solutions of Cauchy problem for Lotka-Volterra system to more general system (1) with condition (2).

2. Main Result

In their researches, Chueh, Con lay and Smoller have considerably developed the invariant region method [3] which was introduced by Weighberger in [4]. In order to deal with the existence, uniqueness and asymptotic behavior of solutions, we give an outline of the invariant region argument.

Definition 1 [1] *A closed subset* $\sum \subset \boldsymbol{R}^n$ *is called an invariant region for the local solution defined by* (1), (2), *if any solution* $u(t,x)$ *having its initial value* $u(0,x) = u^0(x) \in \sum$ *for all* $x \in \boldsymbol{R}^m$, *satisfies* $u(t,x) \in \sum$ *for all* $t \in [0,\delta)(\delta > 0)$.

Now, we consider region $\sum$ of the form

$$\sum = \bigcup_{i=1}^{k} \{u \in \boldsymbol{R}^n_+ | G_i(u) \leq 0\}$$

where $G_i(u), i = 1, \cdots, k$ are smooth real-valued functions defined on open subset of $\boldsymbol{R}^n_+$, and

$$dG_i = \Big(\frac{\partial G_i}{\partial u_1}, \cdots, \frac{\partial G_i}{\partial u_n}\Big) \neq 0$$

Definition 2 [1] *The smooth function* $G : \boldsymbol{R}^n_+ \to \boldsymbol{R}$ *is called quasi-convex at* u, *if whenever* $dG_u(\eta) = 0$, *then* $d^2G_u(\eta,\eta) \geq 0$, *where*

$$d^2G(u) = \Big(\frac{\partial^2 G(u)}{\partial u_i \partial u_j}\Big)_{n\times n}$$

According to the preceding definitions, the following result was given in [1].

Lemma 1 *Consider the above* $\sum$, *and suppose that for all* $t \in \boldsymbol{R}_+$ *and for every* $u^0 \in \partial \sum$ *(so* $G_i(u^0) = 0$ *for some* i*), the following conditions hold:*

1) dG_i *at* u^0 *is a left eigenvector of* D, *for all* $x \in \boldsymbol{R}^m$,

2) *If* $dG_i D = \mu dG_i$, *with* $\mu \neq 0$, *the* G_i *is quasi-convex at* u^0,

3) $dG_i(f) < 0$ *at* u^0, *for all* $t \in \boldsymbol{R}_+$. *Then* $\sum$ *is invariant for* (1).

Obviously, if system (1) admits a bounded invariant region $\sum$, and $u^0(x) \in \sum$ for all $x \in \boldsymbol{R}^m_+$, then the solution of problem (1) and (2) exists for all $t > 0$ and this solution must be bounded.