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GLOBAL SOLVABILITY OF LINEAR DIFFERENTIAL OPERATORS WITH MULTIPLE CHARACTERISTICS

Feng Xueshang
(The 7th Research Division, Beijing University of Aeronautics and Astronautics, Beijing 100083, China)
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Abstract In this paper, we first introduce the irreducible unitary representation of nilpotent Lie groups, then by using the irreducible unitary representation we construct a fundamental solution to a class of left invariant differential operators and thus obtain the global solvability of this kind of operators.
Key Words Irreducible unitary representation; left invariant differential operators; global solvability.
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1. Introduction

It is well-known that there is a lot of works devoted to the study of left invariant operators on nilpotent Lie groups. But the principal part of the operators is always assumed to be elliptic in the generating direction, and some other strong conditions must be added for the operator in order to solve it locally or globally [1–5]. In this paper, by using the complexified irreducible unitary representation we obtain the global solvability for a class of left invariant differential operators, which are not necessarily elliptic in the generating direction.

2. Plancherel Formula and Complexified Representation

Consider Lie group G, whose Lie algebra L admits the following decomposition

$$L = L_1 + L_2$$

Let $\{X_1, \cdots, X_n\}$ and $\{Y_1, \cdots, Y_k\}$ be the bases of L_1 and L_2, respectively. Assume that for each $\eta \in \boldsymbol{R}^k \setminus \{0\}$, there exists on L_1 an antisymmetric bilinear form B_η satisfying the following condition

(H) $\forall \eta \in \boldsymbol{R}^k \setminus \{0\}$, we also denote by B_η the $n \times n$ matrix $(b_{ij}) = (B_\eta[X_i; X_j])$. Suppose that the rank of the kernel of B_η is of constant d.

From the assumption we know that $n = d + 2m$, where $d, m \in N = \{0, 1, 2, \cdots\}$ and for $\eta \neq 0$ fixed there exists an orthogonal transform T_η such that T_η changes $\{X_1, \cdots, X_n\}$ into $\{R_i, U_j, V_j\}, i = 1, \cdots, d; j = 1, \cdots, m$ with the following relations

$$\begin{aligned} &B_\eta[R_i, U_j] = B_\eta[U_i, U_j] = B_\eta[V_i, V_j] = B_\eta[R_i, V_j] = B_\eta[R_i, R_j] = 0 \\ &B_\eta[U_i, V_j] = \delta_{ij}\rho_j(\eta) \end{aligned} \tag{2.1}$$

where $\pm\rho_j(\rho_j > 0)$ are the nonzero eigenvalues of iB_η.

In fact, Lie algebra L of G is isomorphic to the algebra generated by the following vector fields

$$[X_i, X_j] = \sum_{l=1}^{k} A_{ij}^{(l)} Y_l, \qquad i, j = 1, 2, \cdots, n$$

where $A^{(l)} = (A_{ij}^{(l)})$ are antisymmetric $n \times n$ matrices. By Campbell-Hausdorff formula, we know that under the exponential coordinates X_i, Y_l can be written as

$$\begin{cases} X_i = \dfrac{\partial}{\partial x_i} - \dfrac{1}{2}\displaystyle\sum_{l,j} A_{ij}^{(l)} X_j \dfrac{\partial}{\partial y_l}, & i = 1, 2, \cdots, n \\ Y_l = \dfrac{\partial}{\partial y_l}, & l = 1, 2, \cdots, k \end{cases}$$

And $\{X_i\}, \{Y_j\}$ generate L_1, L_2, respectively. The matrix B_η can be obtained in the following way: for $\eta \in L_2^*$ (dual of L_2), define on $L_1 \times L_1$ an antisymmetric bilinear form B_η

$$B_\eta[X, X'] = \eta([X, X']) \quad \text{for } (X, X') \in L_1 \times L_1$$

If η in L_2^* takes the coordinate $(\eta_1, \cdots, \eta_k)$, then the matrix of B_η under $\{X_j\}$ (also denoted by B_η) is

$$B_\eta = \sum_{l=1}^{k} A^{(l)} \eta_l$$

In what follows we derive the irreducible unitary representation and its complexification.

Considering $\xi \in R^d$ as the element of the dual of Span $\{R_1, \cdots, R_d\}$ we can define the irreducible unitary representation of G on $L^2(R^m)$ as follows

$$\begin{aligned} &\pi_{\xi,\eta}\Big(\exp\Big(\sum_{i=1}^{d} r_i R_i + \sum_{j=1}^{m}(u_j U_j + v_j V_j) + \sum_{l=1}^{k} y_l Y_l\Big)\Big) f(s) \\ &= \exp\Big[i(y\eta + r\xi) + i\sum_{j=1}^{m} u_j \rho_j v_j + i\sum_{v_j}^{m} \rho_j s_j\Big] f(s + \sqrt{\rho} u) \end{aligned} \tag{2.2}$$

where $s + \sqrt{\rho}u = (s_1 + \sqrt{\rho_1}u_1, \cdots, s_m + \sqrt{\rho_m}u_m)$. Define

$$\pi(\varphi) = \int_G \varphi(g)\pi(g^{-1})dg \tag{2.3}$$