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THE DIFFRACTION PROBLEM AND VERIGIN PROBLEM OF QUASILINEAR PARABOLIC EQUATION IN DIVERGENCE FORM FOR THE ONE-DIMENSIONAL CASE

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Abstract In this paper, we consider the flow of two immiscible fluids in a one-dimensional porous medium (the Verigin problem) and obtain a quasilinear parabolic equation in divergence form with the discontinuous coefficients. We prove first the existence and uniqueness of locally classical solution of the diffraction problem and then prove the existence of local solution of the Verigin problem.
Key Words Porous medium; discontinuous coefficient; diffraction
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0. Introduction

Since the 1940s, the after-production of petroleum by means of waterflooding has been used extensively to raise the production index. The relevant model for permeability can be idealized mathematically as a free boundary problem. Muskat supposed in 1937 a mathematical model for piston-type driving in [1]. Assuming that the flow moves horizontally and touches the boundary $\Gamma : x = h(t)$, and using the Darcy law and the mass conservation law, Verigin obtained in [2] the parabolic problem with respect to pressure p, called the Verigin problem later on.

The theory about the Verigin problem has been vigorously developed only for the one-dimensional case. The linear Verigin problem was studied by Kamynin in [3], [4], by Fulks and Guenther in [5], and by Evans in [6], [7]. Recently, research on quasilinear equations was set about by Meirmanov in [8] and Liang Jing in [9].

In general case, the free boundary is fixed first. And the problem with discontinuous coefficients to be considered first is called the diffraction problem. It was studied by Oleinik with Bernstein method, by Ladyženskaja with integral estimation, by Kamynin with potential method.

We are concerned with

$$\frac{\partial u_i}{\partial t} - \frac{\partial}{\partial x}\Big(k_i(x,t,u_i)\frac{\partial u_i}{\partial x}\Big) = f_i(x,t,u_i), \quad i=1,2$$

$$u_1 = u_2, h'(t) = -g(u_1)k_1(x,t,u_1)\frac{\partial u_1}{\partial x} = -g(u_2)k_2(x,t,u_2)\frac{\partial u_2}{\partial x}, x=h(t)$$

The paper is divided into three sections. Section 1 discusses the uniform estimation for approximate solution under the smoothened coefficients. Section 2 proves the existence and uniqueness of local solution of diffraction problem and discusses the continuous dependence of solution on internal boundary perturbation. Section 3 proves the existence of local solution of the Verigin problem by means of Schauder fixed point theorem.

1. Uniform Estimation for Approximate Solution

Fix $h(t)$ and let $Q_1 = \{(x,t): 0 < x < h(t), 0 < t < T\}$, $Q_2 = \{(x,t): h(t) < x < l, 0 < t < T\}$, $Q_T = Q_1 \cup Q_2$. We are concerned with the following diffraction problem

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \dfrac{\partial}{\partial x}\Big(k_i(x,t,u_i)\dfrac{\partial u_i}{\partial x}\Big) = f_i(x,t,u_i), & (x,t) \in Q_i,\ i=1,2 \\ u_1 = u_2,\ k_1(x,t,u_1)\dfrac{\partial u_1}{\partial x} = k_2(x,t,u_2)\dfrac{\partial u_2}{\partial x}, & x = h(t) \\ u(0,t) = \bar{u}_1(t), u(l,t) = \bar{u}_2(t), u(x,0) = \bar{u}_0(x), h(0) = b \end{cases} \tag{1.0}$$

and we assume

(I) $k_i(x,t,z)$, $f_i(x,t,z) \in C^3([0,l]\times[0,\infty)\times(-\infty,+\infty))$ and there exist constants $\gamma > 0$, $b_1 > 0$, $b_2 > 0$ such that $k_i(x,t,z) \geq \gamma$, $f_i(x,t,z)z \leq b_1 z^2 + b_2$.

(II) $\bar{u}_1, \bar{u}_2 \in C^1[0,T]$, $\bar{u}_0 \in C[0,l]$, $k(x,0,\bar{u}_0(x))\bar{u}_{0x} \in C^1[0,l]$ with $\bar{u}_0(0) = \bar{u}_1(0)$, $\bar{u}_0(l) = \bar{u}_2(0)$ and denote $k(x,t,u) = k_i(x,t,u_i)$, $(x,t) \in Q_i$, $i = 1,2$.

(III) $h(t) \in C^1[0,T]$.

Smoothen the coefficients and let

$$k_\varepsilon(x,t,u) = k_1(x,t,u)(1 - H_\varepsilon(x-h(t)) + k_2(x,t,u)H_\varepsilon(x-h(t))$$

$$f_\varepsilon(x,t,u) = f_1(x,t,u)(1 - H_\varepsilon(x-h(t))) + f_2(x,t,u)H_\varepsilon(x-h(t))$$

where $H_\varepsilon(x) = \begin{cases} 0 & \text{for } x \leq -\varepsilon \\ 1 & \text{for } x \geq \varepsilon \end{cases}$, $H_\varepsilon(x) \in C^3(-\infty,+\infty)$. Smoothen initial-boundary value and fix arbitrarily $\alpha \in (0,1)$. There exist $\bar{u}_{0\varepsilon} \in C^{2+\alpha}[0,l]$, $\bar{u}_{i\varepsilon} \in C^{1+\alpha/2}[0,T]$, being smoothened forms of $\bar{u}_0$ and $\bar{u}_i$ respectively and satisfying the compatibility conditions of orders 0 and 1, such that

$$\|k_\varepsilon(x,0,\bar{u}_{0\varepsilon}(x))\bar{u}_{0\varepsilon x}\|_{\alpha,[0,l]} \leq \|k(x,0,\bar{u}_0(x))\bar{u}_{0x}\|_{\alpha,[0,l]}$$

$$\|\bar{u}_{i\varepsilon}\|_{1,[0,T]} \leq C + \|\bar{u}_i\|_{1,[0,T]}, \quad \|\bar{u}_{0\varepsilon}\|_{0,[0,l]} \leq 2\|\bar{u}_0\|_{0,[0,l]}, \quad i=1,2$$

$$\|\bar{u}_{0\varepsilon}\|_{2,[0,b/2]} + \|\bar{u}_{0\varepsilon}\|_{2,[(b+l)/2,l]} \leq 2\|\bar{u}_0\|_{2,[0,b/2]} + 2\|\bar{u}_0\|_{2,[(b+l)/2,l]}$$